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SOME IDENTITIES VALID IN SPECIAL JORDAN ALGEBRAS BUT NOT VALID IN ALL JORDAN ALGEBRAS

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A Jordan algebra is defined by the identities:

$$x \cdot y = y \cdot x, (x \cdot y) \cdot y^2 = (x \cdot y^2) \cdot y \; . \tag{1}$$

The algebra A_J obtained from an associative algebra A on replacing the product xy by $x \cdot y = 1/2(xy + yx)$ is easily seen to be a Jordan algebra. Any subalgebra of a Jordan algebra of this type is called special. It is known from work of Albert and Paige that the kernel of the natural homomorphism from the free Jordan algebra on three generators to the free special Jordan algebra on three generators is nonzero and consequently that there exist three-variable relations which hold identically in any homomorphic image of a special Jordan algebra but which are not consequences of the defining identities (1). Such a relation we shall call an S-identity. It is the purpose of this paper to establish that the minimum possible degree for an S-identity is 8 and to give an example of an S-identity of degree 8. In the final section we use an S-identity to give a short proof of the main theorem of Albert and Paige in a slightly strengthened form.

NOTATION. The product in a Jordan algebra will be denoted by a dot, thus $a \cdot b$, and $\{abc\}$ will denote the Jordan triple product

$$\{abc\} = a \cdot (b \cdot c) - b \cdot (c \cdot a) + c \cdot (a \cdot b) \; . \tag{2}$$

Unbracketed products $a_1 \cdot a_2 \cdot \cdots \cdot a_n$ will denote left-normed products i.e. $(\cdots((a_1 \cdot a_2 \cdot) \cdot a_3) \cdot \cdots \cdot a_n)$. When working in a special Jordan algebra we shall use juxtaposition, thus ab, to denote the product in the underlying associative algebra. Then $a \cdot b = 1/2(ab + ba)$ and $2\{abc\} = abc + cba$. The free (respectively free special) Jordan algebra on n generators, taken as $x_1, \cdots, x_n$ or as x, y, z if $n = 3$, will be denoted by $J^{(n)}$ (respectively $J_0^{(n)}$) and the kernel of the natural homomorphism ν_n (written as ν for $n = 3$) of $J^{(n)}$ onto $J_0^{(n)}$ by K_n. The subspace of $J^{(n)}$ spanned by the monomials of degree n linear in each of the generators will be denoted by L_n. The underlying associative algebra for $J_0^{(n)}$ is the free associative algebra on n generators: we shall denote this by $A^{(n)}$. Throughout the paper we work over some fixed, but arbitrary, field of characteristic not two.

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