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SOME CHARACTERIZATIONS OF EXPONENTIAL-TYPE DISTRIBUTIONS

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Let $\mathscr{f} = \{f(x; \delta) = \exp[x\delta + q(\delta)], \delta \in (a, b)\}$ be a family of exponential-type probability density-functions (exp. p.d.f.'s) with respect to a σ-finite measure μ. Let $M(t; \delta), a - \delta < t < b - \delta$, denote the moment generating function (m.g.f.) corresponding to $f(x; \delta) \in \mathscr{f}$, and let $c(t; \delta) = \ln M(t; \delta) = \sum_{k=1}^{\infty} \lambda_k(\delta)t^k/k!$ be the cumulative generating function. The main results pertain to characterizations of certain exp. p.d.f.'s in terms of the cumulants $\lambda_k(\delta)$. First, it is shown that if $M(t; \delta_0)$ is the m.g.f., respectively, of a degenerate, Poisson, or normal law for some $\delta_0 \in (a, b)$, then $M(t; \delta)$ is the m.g.f. of the given law for all $\delta \in (a, b)$, and that infinite divisibility (inf. div) of $M(t; \delta_0)$ for some δ_0 implies inf. div. for all δ. Further, it is shown that if $\varphi(t)$ is a nondegenerate, inf. div. characteristic function (ch. f.) with finite fourth cumulant λ_4, then $\lambda_4 = 0$ if and only if $\varphi(t)$ is the ch.f. of a normal law, while if $\lambda_4 = a\lambda_3 = a^2\lambda_2 \neq 0$, then $\varphi(t)$ is the ch.f. of a Poisson law. Combining these results, it follows that if $M(t; \delta_0)$ is inf. div., and nondegenerate, with $\lambda_4(\delta_0) = 0$, then $M(t; \delta)$ is the m.g.f. of a normal law for all $\delta \in (a, b)$. A similar result characterizes the Poisson law. Finally, it is proved that the normal law is the unique exp. p.d.f. which is symmetric.

An exponential-type family of distributions is defined by probability densities of the form

$$f(y; \delta) = \exp[y\delta + q(\delta)], \qquad a < \delta < b \tag{1}$$

with respect to a σ-finite measure μ over a Euclidean sample space $(\mathfrak{X}, \mathfrak{A})$. It is known ([1], p. 51) that the set of parameter points δ such that $\int \exp[\delta y]d\mu(y) < \infty$, is an interval (finite or not). The binomial, Poisson, normal, gamma, and negative binomial disiributions provide familiar examples of exponential-type distributions.

A few structural properties for this family are considered. Section 2 contains some useful lemmas which are applied in § 3 to obtain some characterizations of the Poisson and normal distributions.

2. Some lemmas. Patil [3] has shown that a collection of d.f.'s $\{F(x; \delta): \delta \in (a, b)\}$ is of exponential-type if and only if the

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