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MINIMIZATION OF FUNCTIONS HAVING LIPSCHITZ CONTINUOUS FIRST PARTIAL DERIVATIVES

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A general convergence theorem for the gradient method is proved under hypotheses which are given below. It is then shown that the usual steepest descent and modified steepest descent algorithms converge under the some hypotheses. The modified steepest descent algorithm allows for the possibility of variable stepsize.

For a comparison of our results with results previously obtained, the reader is referred to the discussion at the end of this paper.

Principal conditions. Let f be a real-valued function defined and continuous everywhere on E^n (real Euclidean n-space) and bounded below E^n. For fixed $x_0 \in E^n$ define $S(x_0) = \{x : f(x) \leqq f(x_0)\}$. The function f satisfies: condition I if there exists a *unique* point $x^* \in E^n$ such that $f(x^*) = \inf_{x \in E^n} f(x)$; Condition II at x_0 if $f \in C^1$ on $S(x_0)$ and $\nabla f(x) = 0$ for $x \in S(x_0)$ if and only if $x = x^*$; Condition III at x_0 if $f \in C^1$ on $S(x_0)$ and ∇f is Lipschitz continuous on $S(x_0)$, i.e., there exists a Lipschitz constant $K > 0$ such that $|\nabla f(y) - \nabla f(x)| \leqq K|y - x|$ for every pair $x, y \in S(x_0)$; Condition IV at x_0 if $f \in C^1$ on $S(x_0)$ and if $r > 0$ implies that $m(r) > 0$ where $m(r) = \inf_{x \in S_r(x_0)} |\nabla f(x)|$, $S_r(x_0) = S_r \cap S(x_0)$, $S_r = \{x : |x - x^*| \geqq r\}$, and x^* is any point for which $f(x^*) = \inf_{x \in E^n} f(x)$. (If $S_r(x_0)$ is void, we define $m(r) = \infty$.)

It follows immediately from the definitions of Conditions I through IV that Condition IV implies Conditions I and II, and if $S(x_0)$ is bounded, then Condition IV is equivalent to Conditions I and II.

2. The convergence theorem. In the convergence theorem and its corollaries, we will assume that f is a real-valued function defined and continuous everywhere on E^n, bounded below on E^n, and that Conditions III and IV hold at x_0.

THEOREM. *If $0 < \delta \leqq 1/4K$, then for any $x \in S(x_0)$, the set*

$$(1) \quad S^*(x, \delta) = \{x_\lambda : x_\lambda = x - \lambda \nabla f(x),\ \lambda > 0,\ f(x_\lambda) - f(x) \leqq -\delta\, |\nabla f(x)|^2\}$$

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