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# FACTORIZATIONS OF $p$-SOLVABLE GROUPS

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**The object of this paper is to put in relief one of the ideas which has been very helpful in studying simple groups, viz. using factorizations of $p$-solvable groups to obtain information about the subgroups of a simple group which contain a given $S_p$-subgroup. Since the idea is so simple, it seems to deserve a simple exposition.**

The group $\mathbf{J}(\mathfrak{X})$ was introduced in [3]. In this paper, $\mathbf{J}(\mathfrak{X})$ is again used, together with a similarly defined group, to obtain factorizations of some $p$-solvable groups which are of relevance in the study of simple groups.

As in [3], $m(\mathfrak{X})$ denotes the minimal number of generators of the finite group $\mathfrak{X}$, and $d(\mathfrak{X}) = \max\{m(\mathfrak{A})\}$, $\mathfrak{A}$ ranging over all the abelian subgroups of $\mathfrak{X}$. For each nonnegative integer $n$, let $\mathbf{J}_n(\mathfrak{X}) = \langle \mathfrak{A} \mid \mathfrak{A}$ is an abelian subgroup of $\mathfrak{X}$ with $m(\mathfrak{A}) \geqq d(\mathfrak{X}) - n\rangle$. Thus $\mathbf{J}_0(\mathfrak{X}) = \mathbf{J}(\mathfrak{X})$ and $\mathbf{J}_k(\mathfrak{X}) = \mathfrak{X}$ whenever $k \geqq d(\mathfrak{X}) - 1$. Also $\mathbf{J}_n(\mathfrak{X}) \subseteqq \mathbf{J}_{n+1}(\mathfrak{X})$ for $n = 0, 1, \cdots$.

THEOREM 1. *Suppose* $\mathfrak{G}$ *is a* $p$*-solvable finite group,* $p$ *is a prime, and* $\mathfrak{G}_p$ *is a* $S_p$*-subgroup of* $\mathfrak{G}$*. Suppose also that* $\mathbf{O}_{p'}(\mathfrak{G}) = 1$ *and that one of the following holds*:

(a) $p \geqq 5$.

(b) $p = 3$ *and* $SL(2, 3)$ *is not involved in* $\mathfrak{G}$.

(c) $p = 2$ *and* $SL(2, 2)$ *is not involved in* $\mathfrak{G}$.

*Let* $\mathfrak{H} = \bigcap_{G \in \mathfrak{G}} \mathbf{C}_{\mathfrak{G}}(\mathbf{Z}(\mathfrak{G}_p))^G$. *Then* $\mathfrak{G} = \mathfrak{H} \cdot \mathbf{N}_{\mathfrak{G}}(\mathbf{J}(\mathfrak{G}_p))$ *and if* $p \geqq 5$, *then* $\mathfrak{G} = \mathfrak{H} \cdot \mathbf{N}_{\mathfrak{G}}(\mathbf{J}_1(\mathfrak{G}_p))$. *In particular,* $\mathfrak{G} = \mathbf{C}_{\mathfrak{G}}(\mathbf{Z}(\mathfrak{G}_p)) \cdot \mathbf{N}_{\mathfrak{G}}(\mathbf{J}(\mathfrak{G}_p))$.

*Proof.* Let $\mathfrak{W}_1 = \mathbf{Z}(\mathfrak{G}_p)^{\mathfrak{G}}$, $\mathfrak{W} = \Omega_1(\mathfrak{W}_1)$. Then $\mathfrak{H} = \mathbf{C}_{\mathfrak{G}}(\mathfrak{W}_1)$ and $\mathfrak{H} = \mathbf{O}_p(\mathfrak{G} \bmod \mathfrak{H})$. If $p \geqq 5$, then since $\mathbf{J}(\mathfrak{G}_p)$ char $\mathbf{J}_1(\mathfrak{G}_p)$, it suffices to show that $\mathbf{J}_1(\mathfrak{G}_p) \subseteqq \mathfrak{H}$, while if $p \leqq 3$, it suffices to show that $\mathbf{J}(\mathfrak{G}_p) \subseteqq \mathfrak{H}$.

Suppose the theorem is false and $\mathfrak{G}$ is a minimal counterexample. Let $\mathfrak{A}$ be an abelian subgroup of $\mathfrak{G}_p$, $\mathfrak{A} \not\subseteqq \mathfrak{H}$, and $m(\mathfrak{A}) \geqq d(\mathfrak{G}_p) - \delta$, where $\delta = 0$ if $p \leqq 3$ and $\delta = 1$ if $p \geqq 5$. Let $\mathfrak{K} = \mathbf{O}_{p'}(\mathfrak{G} \bmod \mathfrak{H})$, $\mathfrak{L} = \mathfrak{K}\mathfrak{A}$. Since $\mathfrak{G}_p \cap \mathfrak{L}$ is a $S_p$-subgroup of $\mathfrak{L}$, it follows that the theorem is violated in $\mathfrak{L}$, so by induction, $\mathfrak{L} = \mathfrak{G}$. Minimality of $\mathfrak{G}$ forces $\mathfrak{A}/\mathfrak{A} \cap \mathfrak{H}$ to be cyclic and forces $\mathfrak{K}/\mathfrak{H}$ to be a special $q$-group. On the other hand, since $m(\mathfrak{A}) \geqq d(\mathfrak{G}_p) - \delta$, it follows that $|\mathfrak{W} : \mathfrak{W} \cap \mathfrak{A}| \leqq p^{1+\delta}$. If $p \geqq 5$, Theorem B of Hall-Higman [2] yields a contradiction, while if $p \leqq 3$,

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