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ON THE REGIONS BOUNDED BY HOMOTOPIC CURVES

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We give a short proof of the following theorem of H. I. Levine [2]:

Let $\mathscr{A}$ and $\mathscr{B}$ be nonintersecting simple closed curves on an orientable surface S (compact or not).

(i) If $\mathscr{A}$ is homotopic to zero, then $\mathscr{A}$ bounds a closed disk in S.

(ii) If $\mathscr{A}$ and $\mathscr{B}$ are freely homotopic but not homotopic to zero then $\mathscr{A} \cup \mathscr{B}$ bounds a closed cylinder in S.

The proof is based on some elementary properties of covering surfaces and the Jordan-Schönflies Theorem for planar surfaces and is as follows. Let H be the cyclic subgroup of $\pi_1(S)$ generated by $\mathscr{A}$. By a standard construction (see [1]) we may form a covering surface $\tilde{S} \xrightarrow{\pi} S$ of S having the following properties: (1) $\pi_1(\tilde{S}) \approx H$; (2) we distinguish a point $O \in S$ and a point $\tilde{O} \in \tilde{S}$ lying over O—then if $\mathscr{D}$ is a closed curve in S through O and $\tilde{\mathscr{D}}$ is the lift of $\mathscr{D}$ to $\tilde{S}$ *which passes through* $\tilde{O}$, the curve $\tilde{\mathscr{D}}$ is closed in $\tilde{S}$ if and only if $\mathscr{D} \in H$.

There exist *closed* curves $\tilde{\mathscr{A}}_1$ and $\tilde{\mathscr{B}}_1$ in $\tilde{S}$ which are lifts of $\mathscr{A}$ and $\mathscr{B}$. To see this assume that $O \in \mathscr{A}$ so by (2) $\mathscr{A}$ lifts to a closed curve through $\tilde{O}$. Since $\mathscr{A}$ and $\mathscr{B}$ are freely homotopic there is an arc γ from $\mathscr{A}$ to $\mathscr{B}$ such that $\mathscr{A} \sim \gamma\mathscr{B}\gamma^{-1}$. Then $\gamma\mathscr{B}\gamma^{-1}$ has at least one lift to a closed curve in $\tilde{S}$; hence so does $\mathscr{B}$.

LEMMA. *If $\tilde{R}$ is a compact region in $\tilde{S}$, π is one-to-one on $\partial\tilde{R}$, and π (int $\tilde{R}) \cap \pi(\partial\tilde{R}) = \emptyset$, then π maps $\tilde{R}$ homeomorphically onto $\pi(\tilde{R})$.*

Proof. The hypotheses imply that the region $\tilde{R}_0 \equiv \text{int}\,\tilde{R}$ is an unlimited covering surface of $\pi(\tilde{R}_0)$. We show that every point of $\pi(\tilde{R}_0)$ is covered exactly once. For suppose $\tilde{p}_1, \tilde{p}_2$ are two points over $p \in \pi(\tilde{R}_0)$. Let γ be an arc in $\pi(\tilde{R}_0)$ from p to $\pi(\partial\tilde{R})$ and let $\tilde{\gamma}_1, \tilde{\gamma}_2$, be arcs over γ from $\tilde{p}_1$ and $\tilde{p}_2$. Then by our hypotheses $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ must intersect on $\partial\tilde{R}$, and the trivial curve $\gamma - \gamma$ in S lifts to the curve $\tilde{\gamma}_1 - \tilde{\gamma}_2$ in $\tilde{S}$, a contradiction.

To prove (i) above, we may assume that $\tilde{S}$, the universal covering surface of S, is a disk. Since $\mathscr{A} \sim 1$, every lift of $\mathscr{A}$ to $\tilde{S}$ is a

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