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AN APPLICATION OF A FAMILY HOMOTOPY EXTENSION THEOREM TO ANR SPACES

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The first of the writers, on p. 206 of Introduction to the Theory of Block Assemblages and Related Topics in Topology, NSF Research Report, University of Kansas, 1956, defined a clean-cut pair to be any pair (X, A) in which X is a metrizable space, A is a closed subset of X, A is a strong deformation neighborhood retract of X, and $X - A$ is an ANR. It is shown in the present paper that for each clean-cut pair (X, A), X is an ANR if and only if A is an ANR. A consequence is that for each locally step-finite clean-cut block assemblage (cf. the report cited above), the underlying space is an ANR. One of the central tools is a family homotopy extension theorem.

Consider a topological space X and a set $A \subset X$.

Suppose $A \subset N \subset X$. A *strong deformation retraction in X of N onto A* is a retraction r of N onto A such that there is a homotopy $H: N \times I \to X$ between the identity map on N and r which leaves A pointwise fixed at each stage. Also, A is a *strong deformation retract in X of N* if and only if there is a strong deformation retraction in X of N onto A. (These definitions are handled more generally in [4, pp. 109–111].) A is a *strong deformation neighborhood retract of X* if and only if for each neighborhood U of A in X there is a neighborhood V of A in U such that A is a strong deformation neighborhood retract in U of V. (This definition is taken from [4, p. 127].) It is observed in [4, pp. 127–128] that A is a strong deformation neighborhood retract of X if and only if A is a strong deformation retract in X of some neighborhood of A.

By an ANR we shall mean an ANR relative to the class of all metrizable spaces.

In [4, p. 206] the pair (X, A) is defined to be *clean-cut* if and only if X is metrizable, A is a closed subset of X, A is a strong deformation neighborhood retract of X, and $X - A$ is an ANR.

In §2 it will be shown that if (X, A) is a clean-cut pair, then X is an ANR if and only if A is an ANR. The "only if" part is trivial. The proof of the "if" part will be based on the usual LC characterization of an ANR and the following proposition from [4, p. 181] (the hypothesis there that $\{X_j\}_{j \in J}$ covers X is inessential since X and K may be added to the respective families).

PROPOSITION 1.1. Suppose that X is a topological space and that

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