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INTEGRAL EQUATIONS AND PRODUCT INTEGRALS

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H. S. Wall, J. S. MacNerney and T. H. Hildebrandt have shown interdependencies between the equations

$$f(x) = {}_a\prod^x (1 + dg) \qquad \text{and} \qquad f(x) = 1 + \int_a^x f dg\,;$$

this paper extends and consolidates some of their results. Let S be a linearly ordered set, R be a normed ring, and OA^0 and OM^0 be classes of functions G from $S \times S$ to R for which

$$\int_a^b \left| G(I) - \int_I G \right| = 0$$

and

$$\int_a^b | [1 + G(I)] - \prod_I (1 + G) | = 0\,,$$

respectively. We show the following. If G has bounded variation, $G \in OA^0$ if and only if $G \in OM^0$. For some rings, the existence of $\int_a^b G(I)$ and ${}_a\prod^b [1 + G(I)]$ imply that $G \in OA^0$ and OM^0, respectively. This is used to prove a product integral solution of integral equations such as

$$f(x) = f(a) + (RL)\int_a^x (fG + Hf)\,,$$

where f is a function from S to R and G and H are functions from $S \times S$ to R. Then these results are used (a) to show that each nonsingular $m \times m$ matrix of complex numbers has n distinct nth roots, (b) to show that, with some restrictions, $\sum_{i=1}^\infty A_i$ exists if and only if $\prod_{i=1}^\infty (1 + A_i)$ exists and (c) to find solutions of integrals equations such as

$$f(x) = f(a) + \int_a^x f^n dg\,.$$

In his recent paper, Integral Equations and Semigroups [7], J. S. MacNerney develops some of the interdependencies between additive and multiplicative integration processes for rings, defines two classes OA and OM of functions V and W such that the integral-like formulas

$$V(a, b) = \int_a^b (W - 1) \qquad \text{and} \qquad W(a, b) = {}_a\prod^b (1 + V)$$

are mutually reciprocal, shows a one-to-one correspondence between the

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