PACIFIC JOURNAL OF MATHEMATICS
Vol. 16, No. 2, 1966

# SEMIGROUPS, PRESBURGER FORMULAS, AND LANGUAGES

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**An interesting class of subsets of lattice points in $n$-space arises naturally in the mathematical theory of (context free) languages. This is the class of semilinear subsets, a subset of lattice points being semilinear if it is the finite union of cosets of finitely generated sub-semigroups of the set of all lattice points with nonnegative coordinates.**

**The family of semilinear sets is here shown to be equivalent to the family of sets defined by modified Presburger formulas. A characterization of those semilinear sets which correspond to languages is then given. Finally, using the two preceding results and the known decidability of the truth of a modified Presburger sentence, a decision procedure is given for determining whether an arbitrary linear set corresponds to a language.**

The class of semilinear sets, first considered in [3] was extensively studied in connection with the theory of bounded languages [1]. In [1] it was shown that the class of semilinear sets is closed with respect to Boolean operations. A consequence of these techniques (in particular, of the proof of Theorem 6.1 of [1]) is that the intersection of two finitely generated sub-semigroups of nonnegative lattice points in $n$-space is itself a finitely generated sub-semigroup.

The definition of a semilinear set as a finite union is an "internal" description of the set. More precisely, a semilinear set is defined by a finite set of nonnegative lattice points (called *constants*) to each of which is associated a finite set of nonnegative lattice points (called *periods*). The semilinear set is the set generated by adding to each constant an arbitrary finite sequence of its associated periods (allowing repetitions of the same period in the sequence).

Another class of subsets of nonnegative lattice points is defined by the modified Presburger formulas. This class is also closed with respect to Boolean operations [5]. The subsets in this class are defined by an "external" description. More precisely, each set in the class is defined as the extension of a modified Presburger formula with $n$ free variables, i.e., the set of all $n$-tuples of nonnegative integers satisfying the given formula.

Section 1 contains a proof that the family of semilinear sets is

Received October 9, 1964. This work was supported in part by the Air Force Cambridge Research Laboratories, Office of Aerospace Research, under Contract AF 19 (628)-485.