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A THEOREM ON THE ACTION OF ABELIAN UNITARY GROUPS

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Given an abelian unitary group G acting on the Hilbert space $\mathscr{H}$, let $\mathscr{A}$ be the C^*-algebra generated by G and let $\sigma(\mathscr{A})$ denote the maximal ideal space of this algebra. There is a natural injection α of $\sigma(\mathscr{A})$ into the compact character group Γ of the discrete group G. What conditions on G will ensure that α be a topological homeomorphism of $\sigma(\mathscr{A})$ on Γ?

The action of G is said to be nondegenerate if, for every finite subset F of G, there exists a vector $\xi \neq 0$ in $\mathscr{H}$ such that $U\xi \perp V\xi$ for every pair U, V of distinct elements of F. Theorem 1 contains the following answer to our question; in order that α map $\sigma(\mathscr{A})$ onto Γ, it is necessary and sufficient that the action of G be nondegenerate.

To be more explicit, α is the mapping that merely restricts every complex homomorphism $\omega \in \sigma(\mathscr{A})$ to the group G. α is automatically continuous by definition of the topologies involved, and it is one-to-one because a bounded linear functional on $\mathscr{A}$ is completely determined by its values on G, the latter being a fundamental set in $\mathscr{A}$. α will be a homeomorphism, therefore, provided only that every character in Γ be the image of something in $\sigma(\mathscr{A})$.

Our interest in this problem arose out of a desire to characterize, in terms of action, when the spectrum of a unitary operator will fill out the unit circle. An appropriately translated version of Theorem 1 gives the following criterion: the spectrum of a unitary U is the entire circle if, and only if, for every integer $n \geqq 1$ there exists a nonzero vector ξ such that ξ, $U\xi$, $\cdots$, $U^n\xi$ are mutually orthogonal.

Versions of the sufficiency part of this problem have been considered before. Some time ago, Kodaira and Kakutani (6) showed essentially that α is onto Γ when G is the discrete unitary group determined by the left regular representation of a locally compact abelian group in its own L_2 space. Their proof involves the Plancherel theorem and is not available in this context. Recently, A. Ionescu-Tulcea (5) has shown that if U is the unitary operator induced in L_2 of a σ-finite measure space by a nonperiodic invertible measure preserving transformation, then the spectrum of U is the entire unit circle.

2. Examples. First, let us note that the definition of nondegener-

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