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INVARIANT MEANS ON TOPOLOGICAL SEMIGROUPS

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This paper is concerned with the existence and structure of invariant means on the space $C(S)$ of *all* (including unbounded) continuous real-valued functions on a topological semigroup S. The main result is that for realcompact semigroups every left invariant mean (if any exist) arises as an integral over a compact left invariant subset of S. The question of existence of noncompact group G such that $C(G)$ admits a left invariant mean is also considered. If G is a realcompact (or discrete, or locally compact abelian) group, then $C(G)$ admits a left invariant mean only if G is compact.

By a *topological semigroup* we mean a semigroup S endowed with a Hausdorff topology for which the mapping $(x, y) \rightarrow xy$ of $S \times S$ into S is continuous. We denote by $C(S)$ the space of *all* continuous real-valued functions on S (not necessarily bounded). If $a \in S$ and $f \in C(S)$, then ${}_af$ and f_a will denote those functions on S whose values at $x \in S$ are $f(ax)$ and $f(xa)$ respectively. Obviously ${}_af$ and f_a are elements of $C(S)$. A *left invariant mean* on $C(S)$ is a nonnegative linear functional M on $C(S)$ such that that $M(1) = 1$ and $M({}_af) = M(f)$ for all $a \in S$ and $f \in C(S)$. *Right invariant means* are defined similarly, replacing ${}_af$ by f_a. A functional M that is both a left invariant mean and a right invariant mean is called a two-sided invariant mean.

The purpose of this to study the structure of invariant means on $C(S)$. This problem differs from the usual problem of finding invariant means for semigroups (see reference [2]), which concerns invariant means on the space $C^*(S)$ of all *bounded* continuous real-valued functions on S. Of course S is pseudocompact, then $C(S) = C^*(S)$ and the two problems are the same. Our main result is to the effect that, for a large class of semigroups S, every left invariant mean (if any exist) arises in a particularly simple way; namely, as an integral over a compact left invariant subset of S. In the final section we consider the question of whether there exist noncompact groups G such that $C(G)$ admits a left invariant mean. A negative answer is obtained for noncompact groups of various types.

In this section we establish some terminology and preliminary facts needed for the statement and proof of the main theorem.

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