PACIFIC JOURNAL OF MATHEMATICS
Vol. 16, No. 3, 1966

INVARIANT SUBSPACES OF POLYNOMIALLY COMPACT OPERATORS

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This paper is a comment on the solution of an invariant subspace problem by A. R. Bernstein and A. Robinson [2]. The theorem they prove can be stated as follows: if A is an operator on a Hilbert space H of dimension greater than 1, and if p is a nonzero polynomial such that $p(A)$ is compact, then there exists a nontrivial subspace of H invariant under A. ("Operator" means bounded linear transformation; "Hilbert space" means complete complex inner product space; "compact" means completely continuous; "subspace" means closed linear manifold; "nontrivial", for subspaces, means distinct from $\{0\}$ and from H.) The Bernstein-Robinson proof has two aspects: it is an ingenious adaptation of the proof by N. Aronszajn and K. T. Smith of the corresponding theorem for compact operators [1], and it makes strong use of metamathematical concepts such as nonstandard models of higher order predicate languages. The purpose of this paper is to show that by appropriate small modifications the Bernstein-Robinson proof can be converted (and shortened) into one that is expressible in the standard framework of classical analysis.

A quick glance at the problem is sufficient to show that there is no loss of generality in assuming the existence of a unit vector e such that the vectors $e, Ae, A^2e, \cdots$ are linearly independent and have H for their (closed linear) span. (This comment appears in both [1] and [2].) The Gram-Schmidt orthogonalization process applied to the sequence $\{e, Ae, A^2e, \cdots\}$ yields an orthonormal basis $\{e_1, e_2, e_3, \cdots\}$ with the property that the span of $\{e, \cdots, A^{n-1}e\}$ is the same as the span of $\{e_1, \cdots, e_n\}$ for each positive integer n. It follows that if $a_{mn} = (Ae_n, e_m)$, then $a_{mn} = 0$ unless $m \leqq n + 1$; in other words, in the matrix of A all entries more than one step below the main diagonal must vanish. The matrix entries of the kth power of A are given by $a_{mn}^{(k)} = (A^k e_n, e_m)$. A straightforward induction argument, based on matrix multiplication, yields the result that $a_{mn}^{(k)} = 0$ unless $m \leqq n + k$, and

$$a_{n+k,n}^{(k)} = \Pi_{1 \leqq j \leqq k} a_{n+j,\, n+j-1}.$$

(With the usual understanding about an empty product having the value 1, the result is true for $k = 0$ also.) This result for powers has an implication for polynomials. If the degree of p (the only polynomial

Received October 10, 1964. Research supported in part by a grant from the National Science Foundation.