PACIFIC JOURNAL OF MATHEMATICS
Vol. 16, No. 3, 1966

SOLUTION OF AN INVARIANT SUBSPACE PROBLEM OF K. T. SMITH AND P. R. HALMOS

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The following theorem is proved.

Let T be a bounded linear operator on an infinite-dimensional Hilbert space H over the complex numbers and let $p(z) \neq 0$ be a polynomial with complex coefficients such that $p(T)$ is completely continuous (compact). Then T leaves invariant at least one closed linear subspace of H other than H or $\{0\}$.

For $p(z) = z^2$ this settles a problem raised by P. R. Halmos and K. T. Smith.

The proof is within the framework of Nonstandard Analysis. That is to say, we associate with the Hilbert space H (which, ruling out trivial cases, may be supposed separable) a larger space, $^*H$, which has the same formal properties within a language L. L is a higher order language but $^*H$ still exists if we interpret the sentences of L in the sense of Henkin. The system of *natural numbers* which is associated with $^*H$ is a nonstandard model of arithmetic, i.e., it contains elements other than the standard natural numbers. The problem is solved by reducing it to the consideration of invariant subspaces in a subspace of $^*H$ the number of whose dimensions is a nonstandard positive integer.

1. Introduction.

We shall prove:

MAIN THEOREM 1.1. *Let T be a bounded linear operator on an infinite-dimensional Hilbert space H over the complex numbers and let $p(z) \neq 0$ be a polynomial with complex coefficients such that $p(T)$ is completely continuous (compact). Then T leaves invariant at least one closed subspace of H other than H or $\{0\}$.*

For $p(z) = z^2$ this settles Problem No. 9 raised by Halmos in [2] and there credited to K. T. Smith. For this case, a first proof was given by one of us (A.R.) while the other (A.R.B.) provided an alternative proof which extends to the case considered in 1.1. The argument given below combines the two proofs, both of which are based on Nonstandard Analysis. The Nonstandard Analysis of Hilbert space was developed previously by A.R. as far as the spectral analysis of completely continuous self-adjoint operators (compare [7]) while A.R.B. has disposed of the spectral theorem for bounded self-adjoint operators

Received July 5, 1964, and in revised form December 10, 1964. The authors acknowledge with thanks the support received from the National Science Foundation (Grant No. GP-1812).