PACIFIC JOURNAL OF MATHEMATICS
Vol. 16, No. 3, 1966

TWO NOTES ON REGRESSIVE ISOLS

JOSEPH BARBACK

This paper deals with regressive functions and regressive isols. It was proven by J. C. E. Dekker in [2] that the collection Λ_R of all regressive isols is not closed under addition. In the first note of this paper we shall given another proof of this fact by considering a new relation, denoted by $\overset{*}{\vee}$, between infinite regressive isols. Let A and B denote infinite regressive isols. The main results established in the first note are:

(1) $A \leqq^* B \Longrightarrow A \overset{*}{\vee} B$, yet not conversely.

(2) $A + B \in \Lambda_R \Longrightarrow A \overset{*}{\vee} B$, yet not conversely.

(3) There exist infinite regressive isols which are not $\overset{*}{\vee}$ related.

(4) Λ_R is not closed under addition.

In addition, the following result is stated.

(5) $A + B \in \Lambda_R \Longrightarrow \min(A, B) \leqq A + B$, yet not conversely.

In the second note we consider the $\leqq^*$ relation between regressive isols. A natural question concerning this relation is whether A $\leqq^* B$, where A and B are regressive isols, is a necessary or a sufficient condition for the sum $A + B$ to be regressive. In the second note we show that this condition is neither necessary nor sufficient.

We shall assume that the reader is familiar with the notations, terminology and main results of [1] and [2].

Preliminaries. Let $\varepsilon = \{0, 1, 2, 3, \cdots\}$ be the set of nonnegative integers (*numbers*). A one-to-one function t_n from ε into ε is *regressive* if there is a partial recursive function $p(x)$ such that $\rho t \subseteqq \delta p$ and $p(t_0) = t_0$, $(\forall n)[p(t_{n+1}) = t_n]$. The function p is a *regressing function* of t_n if p has the following additional properties: $\rho p \subseteqq \delta p$ and $(\forall x)[x \in \delta p \rightarrow (\exists n)[p^{n+1}(x) = p^n(x)]]$. It is known (cf. [1]) that every regressive function has a regressing function. A set is *regressive* if it is finite or the range of a regressive function. A set is *retraceable* if it is finite or the range of a strictly increasing regressive function. Let p be a regressing function of t_n, then the function p^* is defined by: $\delta p^* = \delta p$ and $p^*(x) = (\mu n)[p^{n+1}(x) = p^n(x)]$. It follows that p^* is a partial recursive function and $(\forall n)[p^*(t_n) = n]$.

Let s_n and t_n be two one-to-one functions from ε into ε. Then $s_n \leqq^* t_n$, if there is a partial recursive function f such that

$$(1) \qquad \rho s \subseteqq \delta f \quad \text{and} \quad (\forall n)[f(s_n) = t_n]\,.$$

Also, s_n and t_n are said to be *recursively equivalent* (denoted $s_n \simeq t_n$)

Received October 5, 1964. Most of the results contained in this paper were obtained while the author was a student of Professor J. C. E. Dekker at Rutgers University. Research on this paper was supported under NSF Grant GP-266.