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A TRANSPLANTATION THEOREM FOR ULTRASPHERICAL COEFFICIENTS

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Let $f(\theta)$ be integrable on $(0, \pi)$ and define

$$a_n = \int_0^\pi f(\theta)\cos n\theta\, d\theta\ , \qquad b_n = n^{1/2}\int_0^\pi f(\theta)P_n(\cos\theta)(\sin\theta)^{1/2}d\theta$$

where $P_n(x)$ is the Legendre polynomial of degree n. Then

$$(1) \qquad c \leqq \sum_{n=0}^{\infty} |a_n|^p(n+1)^\alpha \Big/ \sum_{n=0}^{\infty} |b_n|^p(n+1)^\alpha \leqq C$$

for $1 < p < \infty$, $-1 < \alpha < p-1$, where C and c depend on p and α but not on f. From this we obtain a form of the Marcinkiewicz multiplier theorem for Legendre coefficients. Also an analogue of the Hardy-Littlewood theorem on Fourier coefficients of monotone coefficients is obtained. In fact, any norm theorem for Fourier functions can be transplanted by (1) to a corresponding theorem for Legendre coefficients.

Actually, the main theorem of this paper deals with ultraspherical coefficients and (1) is just a typical special case, which is stated as above for simplicity.

Let $P_n^\lambda(x)$ be defined by $(1 - 2rx + r^2)^{-\lambda} = \sum_{n=0}^\infty P_n^\lambda(x)r^n$ for $\lambda > 0$. The functions $P_n^\lambda(\cos\theta)$ are orthogonal on $(0, \pi)$ with respect to the measure $(\sin\theta)^{2\lambda}d\theta$ and

$$(1) \qquad \int_0^\pi [P_n^\lambda(\cos\theta)]^2(\sin\theta)^{2\lambda}d\theta = \frac{\Gamma(n+2\lambda)\Gamma(1/2)\Gamma(\lambda+1/2)}{n!(n+\lambda)\Gamma(\lambda)\Gamma(2\lambda)} = [t_n^\lambda]^{-2}\ .$$

Observe that $t_n^\lambda = An^{1-\lambda} + O(n^{-\lambda})$ where A will denote a constant whose numerical value is of no interest to us. For simplicity we set $\varphi_n^\lambda(\theta) = t_n^\lambda P_n^\lambda(\cos\theta)(\sin\theta)^\lambda$. The functions $\{\varphi_n^\lambda(\theta)\}_{n=0}^\infty$ form a complete orthonormal sequence of functions on $(0, \pi)$ which for $\lambda=1$ reduce to $\{A\sin(n+1)\theta\}_0^\infty$. Also $\lim_{\lambda\to 0}\varphi_n^\lambda(\theta) = A\cos n\theta$ so the functions $\varphi_n^\lambda(\theta)$ are generalizations of the trigonometric functions which are used in classical Fourier series. For uniformity we define $\varphi_n^0(\theta) = (2/\pi)^{1/2}\cos n\theta$. Later we shall state an asymptotic formula for $\varphi_n^\lambda(\theta)$ which shows another close connection with trigonometric functions. In essence it says that $\varphi_n^\lambda(\theta)$ looks like $\cos[(n+\lambda)\theta - \pi(\lambda/2)]$. All of the facts about φ_n^λ that are quoted without reference are in [15]. Since $\varphi_n^\lambda(\theta)$ are a bounded orthonormal sequence we may consider their Fourier coefficients. Let $f \in L^1(0, \pi)$ and define

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