PACIFIC JOURNAL OF MATHEMATICS
Vol. 17, No. 1, 1966

TOPOLOGICAL METHODS FOR NON-LINEAR ELLIPTIC EQUATIONS OF ARBITRARY ORDER

FELIX E. BROWDER

Consider a strongly elliptic nonlinear partial differential equation (e): $F(x, u, Du, \cdots, D^{2m}u) = 0$, of order $2m$ on a bounded, smoothly bounded subset Ω of R^n. For second-order operators, Leray and Schauder, using the theory of the topological degree for completely continuous displacements of a Banach space, showed that the existence of solutions of the Dirichlet problem for (e) could be proved under the assumption of suitable a-priori bounds for solutions of the type of (e). In the present paper, using precise results on the solutions of linear elliptic differential operators with Hölder continuous coefficients as well as a variant of the Leray-Schauder method, we extend this result to equations of arbitrary even order. We also obtain results on uniqueness in the large under hypotheses of local uniqueness.

Theorem 1 is our general result of Leray-Schauder type for the most general sort of strongly elliptic nonlinear equation. Its proof is based upon Theorems 2 and 3 which concern equations for which one has local uniqueness of solutions. Theorem 2, which extends a result of Schauder [16] for second order equations, asserts the solvability of the equation $F(u) = f$ for f near f_0 with u near u_0 if the solution is locally unique. Under similar hypotheses and an additional a priori bound, Theorem 3 asserts the existence and uniqueness of the solution for all f. Theorem 4 and 5 specialize Theorem 1 with a drastic simplification of hypotheses to quasi-linear equations of order $2m$ and to nonlinear second-order equations. Theorem 4, in particular, gives a simple and very general extension of the Leray-Schauder method as given in [9] for quasi-linear equations of second order.

The writer is indebted to Stephen Smale for a number of conversations which stimulated his interest in giving a systematic treatment of the Leray-Schauder theory for general non-linear elliptic equations.

1. Let Ω be a bounded, smoothly bounded open subset of the Euclidean space R^n, Γ its boundary in R^n, $\bar{\Omega}$ its closure in R^n, $(n \geqq 1)$. We denote the general point of Ω by $x = (x_1, \cdots, x_n)$ and for each n-tuple $\alpha = (\alpha_1, \cdots, \alpha_n)$ of nonnegative integers, we set

$$D^\alpha = \prod_{j=1}^{n} \left(\frac{\partial}{\partial x_j}\right)^{\alpha_j}, \qquad |\alpha| = \sum_{j=1}^{n} \alpha_j .$$

Received October 23, 1964.