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WHICH WEIGHTED SHIFTS ARE SUBNORMAL

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Let H be a Hilbert space with orthonormal basis $\{f_j\}_{j=1}^{\infty}$. If the operator T is defined on H by $Tf_i = a_i f_{i+1}$ for $i = 1, 2, \cdots$, where $|a_i| \leqq |a_{i+1}| \leqq M$ for $i = 1, 2, \cdots$, then T will be called a monotone shift. The first section of the paper examines some of the elementary properties of such operators.

Every monotone shift is hyponormal. The central portion of the paper aims at discovering which monotone shifts are subnormal. Necessary and sufficient conditions are given in terms of the $\{a_i\}$. These conditions make it easy to show that even the first four coefficients ($a_1 < a_2 < a_3 < a_4$) may "prevent" a shift from being subnormal. However, for any $a_1 < a_2 < a_3$ there does exist a monotone shift with these as its initial terms. In fact, the unique minimal one is constructed.

A complete description is given of subnormal monotone shifts for which $|a_{j_0}| = |a_{j_0+1}|$ for some j_0. The paper concludes with counter-examples constructed from the machinery developed.

We are tacitly assuming that $\lim_{j\to\infty} |a_j|$ exists, i.e., T is a bounded operator. If $|a_j| = |a_{j+1}|$ for $j = 1, 2, \cdots$, then T is (up to unitary equivalence) simply a multiple of the justly famous unilateral shift.

We recall that an operator T on a Hilbert space H is subnormal if it is the restriction of a normal operator to an invariant subspace. The terms "point", "continuous" and "residual spectrum" have their usual meaning and are designated by $\sigma_P(\cdot)$, $\sigma_C(\cdot)$ and $\sigma_R(\cdot)$ respectively.

THEOREM 1. *Let T be a monotone shift on H where $A = \lim_{j\to\infty} |a_j|$, then*
 (i) $\|T\| = A$
 (ii) $\sigma_R(T) = \{z: |z| < A\}$
 (iii) $\sigma_P(T^*) = \{z: |z| < A\}$
 (iv) $\sigma_C(T) = \sigma_C(T^*) = \{z: |z| = A\}$.

Proof. Surely (i) is clear.

For $|z_0| < A$, consider the vector $g = \sum_{n=1}^{\infty} z_0^n b_n f_n$ where $b_k = 1$ if $a_{k-1} = 0$, $a_k \neq 0$ and $b_{n+1} = z_0 b_n / \bar{a}_n$ for $n > k$. Since $|z_0/\bar{a}_n| \leqq r < 1$ for n sufficiently large, $g \in H$. But $(T^* - zI)g = 0$ so g is the desired eigenvector, proving (iii). The relation $b_{n+1} = z_0 b_n / \bar{a}_n$ is necessary which implies the eigenvalue z_0 is of multiplicity one.

For any z, it is clear that $(T - zI)h \neq 0$ for $h \in H$, and $h \neq 0$.

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