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DIVISIBILITY PROPERTIES OF CERTAIN FACTORIALS

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It is well known that multinomial coefficients are integers; i.e., if the integers a_i are nonnegative and $a = \sum_{i=1}^{m} a_i$, then $\prod_{i=1}^{m} (a_i)! \mid a!$. This property may hold good in special cases even though $\sum_{i=1}^{m} a_i > a$. In fact, for each integer $x \geqq 0$, $x!(x+1)! \mid (2x)!$, and it has been asked by Erdos, as a research problem in the 1947 May issue of the Monthly, whether, for a given $c \geqq 1$, there exists an infinity of integers x such that $x!(x+c)! \mid (2x)!$. This problem has been gradually generalized and improved upon by Mordell, Wright, McAndrew, the author, and N. V. Rao. In particular, Rao considers the quotient $Q(x) = ((g(x)+h(x))!)/((g(x)+k)!(h(x))!)$, where k is a positive integer, and $g(x)$ and $h(x)$ are integer coefficient polynomials of positive degree with positive leading coefficients and proves that some multiple of $Q(x)$ is integral infinitely often: a result which includes all the earlier results. In this paper, among other things, this result of Rao has been generalised and improved upon by taking the polynomials over the rationals and by reducing the multiplying factor of $Q(x)$ as obtained by Rao.

Throughout the following i, j, k, r, and n denote positive integral variables and all small letters, unless explicitly mentioned otherwise denote positive integers. As usual, (a, b) and $\{a, b\}$ denote respectively the $G.C.D.$ and $L.C.M.$ of a and b. For any polynomials $X(x)$ and $Y(x)$ (not both zero) over the rationals, $(X(x), Y(x))$ denote their monic $G.C.D.$ over the rationals. m being $\geqq 1$, $t_1, t_2, \cdots, t_m$ are integers each greater than 1. For $1 \leqq i \leqq m$ and $1 \leqq j \leqq t_i$, $f_{ij}(x)$ is a polynomial of positive degree over the rationals with positive leading coefficient; a_{ij} and c_{ij} are nonnegative integers, r_{ij} is a positive rational and k_{ij} is a positive integer. Also, r_i is a nonnegative integer for each i in $1 \leqq i \leqq m$. We use the following symbolism.

$$(1.1) \qquad f_i(x) = \sum_{k=1}^{t_i} f_{ik}(x) \; ; \qquad F_{ij}(x) = \sum_{\substack{k=1 \\ k \neq j}}^{t_i} f_{ik}(x)$$

$$A_i = \sum_{k=1}^{t_i} a_{ik} \; ; \qquad A_{ij} = \sum_{\substack{k=1 \\ k \neq j}}^{t_i} a_{ik}$$

$$R_i = \sum_{k=1}^{t_i} r_{ik} \; ; \qquad R_{ij} = \sum_{\substack{k=1 \\ k \neq j}}^{t_i} r_{ik} \; ;$$

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