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MINIMAL GERSCHGORIN SETS II

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The Gerschgorin Circle Theorem, which yields n disks whose union contains all the eigenvalues of a given $n \times n$ matrix $A = (a_{i,j})$, applies equally well to any matrix $B = (b_{i,j})$ of the set Ω_A of $n \times n$ matrices with $b_{i,i} = a_{i,i}$ and $|b_{i,j}| = |a_{i,j}|$, $1 \leqq i,\ j \leqq n$. This union of n disks thus bounds the entire spectrum $S(\Omega_A)$ of the matrices in Ω_A. The main result of this paper is a precise characterization of $S(\Omega_A)$, which can be determined by extensions of the Gerschgorin Circle Theorem based only on the use of positive diagonal similarity transformations, permutation matrices, and their intersections.

Given any $n \times n$ complex matrix $A = (a_{i,j})$, it is well known that the simplest of Gerschgorin arguments, which depends upon row sums of the moduli of off-diagonal entries of the matrix $X^{-1}AX$, X a positive diagonal matrix, yields the union of n disks which contains all the eigenvalues of A. It is clear that this union of n disks necessarily contains all the eigenvalues of any $n \times n$ matrix in the set Ω_A defined as follows: $B = (b_{i,j}) \in \Omega_A$ if $b_{i,i} = a_{i,i}$, $1 \leqq i \leqq n$, and $|b_{i,j}| = |a_{i,j}|$ for all $1 \leqq i, j \leqq n$, $i \neq j$. Hence, this union of n Gerschgorin disks can be viewed as giving bounds for the entire spectrum $S(\Omega_A) = \{z \mid \det(zI - B) = 0 \text{ for some } B \in \Omega_A\}$ of the set Ω_A.

It is logical to ask to what extent the spectrum $S(\Omega_A)$ can be more precisely determined by extensions of Gerschgorin's original argument [3]. In the previous paper [6], it was shown that

$$\partial G(\Omega_A) \subset S(\Omega_A) \subset G(\Omega_A)\ , \tag{1.1}$$

where $G(\Omega_A)$ is the *minimal Gerschgorin set* deduced from A and $\partial G(\Omega_A)$ is its boundary. The first inclusion of (1.1) states that every point of the boundary $\partial G(\Omega_A)$ of the minimal Gerschgorin set is then an eigenvalue of some $B \in \Omega_A$. We now extend the results of [6] by making use of results of Schneider [4], and Camion and Hoffman [1]. In so doing, we shall *precisely* determine $S(\Omega_A)$.

To begin, let $P_\phi = (\delta_{i,\phi(j)})$ be an $n \times n$ permutation matrix, where ϕ is a permutation of the integers $1 \leqq i \leqq n$ and $\delta_{i,j}$ is the Kronecker delta function, and let $X = \operatorname{diag}(x_1, x_2, \cdots, x_n)$, where $\boldsymbol{x} > \boldsymbol{0}$. Given $B \in \Omega_A$, we define the $n \times n$ matrix $M^\phi(\boldsymbol{x})$ by

$$M^\phi(\boldsymbol{x}) = (X^{-1}BX - \lambda I)P_\phi = (m_{i,j})\ , \tag{1.2}$$

so that

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