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WEAK-STAR GENERATORS OF H^∞

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Let H^∞ denote the algebra of bounded analytic functions in the unit disk $D = \{z: |z| < 1\}$. A function φ in H^∞ is called a generator if the polynomials in φ are weak-star dense in H^∞. The problem to be considered here is that of characterizing the generators of H^∞.

The weak-star topology of H^∞ can be thought of as arising in the following way. By Fatou's theorem, each function ψ in H^∞ has radial limits at almost every point of the unit circle $C = \{z: |z| = 1\}$ and thus gives rise to a bounded measurable function ψ_C on C. The map $\psi \to \psi_C$ sends H^∞ isomorphically and isometrically onto a certain subspace of $L^\infty(C)$; we denote this subspace by $H^\infty(C)$. (We regard C as endowed with normalized Lebesgue measure.) The space $H^\infty(C)$ is the dual of a quotient space of $L^1(C)$ and as such has a weak-star topology (which is simply the topology induced on $H^\infty(C)$ by the weak-star topology of $L^\infty(C)$). Because of the natural correspondence between H^∞ and $H^\infty(C)$, the weak-star topology on the latter induces a topology on the former, and this is what we mean by the weak-star topology of H^∞. The convergent sequences of this topology are easily characterized.

LEMMA 1. *A sequence $\{\psi_n\}_1^\infty$ in H^∞ converges weak-star to the function ψ if and only if it is uniformly bounded and converges to ψ at every point of D.*

Proof. This is of course well-known; however we include a proof for the sake of completeness. To simplify the notation we shall write $\varphi(e^{it})$ in place of $\varphi_C(e^{it})$ for any φ in H^∞ For each point a in D let P_a denote the corresponding Poisson kernel, i.e.,

$$P_a(z) = \frac{1 - |a|^2}{|1 - \bar{a}z|^2}, \quad |z| = 1 .$$

We then have

$$\varphi(a) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(e^{it})P_a(e^{it})dt \tag{1}$$

for all φ in H^∞ and all a in D.

Now suppose the sequence $\{\psi_n\}$ in H^∞ is uniformly bounded and converges to the function ψ at each point of D. Then it follows from

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