PACIFIC JOURNAL OF MATHEMATICS
Vol. 17, No. 3, 1966

CHARACTERIZATION OF THE SUBDIFFERENTIALS OF CONVEX FUNCTIONS

R. T. ROCKAFELLAR

Each lower semi-continuous proper convex function f on a Banach space E defines a certain multivalued mapping ∂f from E to E^* called the subdifferential of f. It is shown here that the mappings arising this way are precisely the ones whose graphs are maximal "cyclically monotone" relations on $E \times E^*$, and that each of these is also a maximal monotone relation. Furthermore, it is proved that ∂f determines f uniquely up to an additive constant. These facts generally fail to hold when E is not a Banach space. The proofs depend on establishing a new result which relates the directional derivatives of f to the existence of approximate subgradients.

Let E be a topological vector space over the real numbers R with dual E^*. Let f be a *proper convex function* on E, i.e., an everywhere-defined function with values in $(-\infty, +\infty]$, not identically $+\infty$, such that

$$f(\lambda x + (1 - \lambda)y) \leqq \lambda f(x) + (1 + \lambda)f(y) \tag{1.1}$$

for all x and y in E when $0 < \lambda < 1$. A *subgradient* of f at $x \in E$ is an $x^* \in E^*$ such that

$$f(y) \geqq f(x) + \langle y - x, x^* \rangle \quad \text{for all} \quad y \in E\,.$$

(This says that $f(x)$ is finite and that the graph of the affine function $h(y) = f(x) + \langle y - x, x^* \rangle$ is a "nonvertical" supporting hyperplane at $(x, f(x))$ to the epigraph of f, which is the convex subset of $E \oplus R$ consisting of all the points lying above the graph of f.) For each $x \in E$, we denote by $\partial f(x)$ the set of all subgradients of f at x, which is a weak* closed convex set in E^*. If $\partial f(x) \neq \emptyset$, f is said to be *subdifferentiable* at x. The *subdifferential* of f is the multivalued mapping (relation) ∂f which assigns the set $\partial f(x)$ to each x.

The notion of subdifferentiability has been developed recently in [3], [6], [8], [11]. Much of the work has concerned the existence of subgradients. It is known, for example, that f is subdifferentiable wherever it is finite and continuous (see [6] or [8]). Results in [3] show among other things that, if E is a Banach space and f is lower semi-continuous (l.s.c.), then the set of points where f is subdiffer-

Received January 7, 1965. This work was supported in part by the Air Force Office of Scientific Research, through a grant at The University of Texas.