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ON THE CHARACTERISTIC ROOTS OF THE PRODUCT OF CERTAIN RATIONAL INTEGRAL MATRICES OF ORDER TWO

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This paper deals with a special case of the following problem: Let A, B be matrices of order n over the rational integers. Compare the algebraic number field generated by the characteristic roots of AB with those generated by A, B.

We let $M(r, s)$ denote the companion matrix of $x^2 + rx + s$, for rational integers r and s, and let $N(r, s) = M(r, s)(M(r, s))'$. Further let $F(M(r, s))$ and $F(N(r, s))$ denote the fields generated by the characteristic roots of $M(r, s)$ and $N(r, s)$ over the rational field, R. This paper is concerned with $F(N(r, s))$, especially in relation to $F(M(r, s))$. The principal results obtained are outlined as follows:

Let S be the set of square-free integers which are sums of two squares. Then $F(N(r, s))$ is of the form $R(\sqrt{c})$, where $c \in S$. Further, $F(N(r, s)) = R$ if and only if $rs = 0$. Suppose $c \in S$. Then there exist infinitely many distinct pairs of integers (r, s) such that $F(N(r, s)) = R(\sqrt{c})$.

Further, if $c \in S$, there exists an infinite sequence $\{(r_n, s_n)\}$ of distinct pairs of integers such that $F(M(r_n, s_n)) = R(\sqrt{c})$ and $F(N(r_n, s_n)) = R(\sqrt{cd_n})$ for some integers d_n such that $(c, d_n) = 1$. If $c \in S$ and c is odd or $c = 2$, there exists an infinite sequence $\{(r'_n, s'_n)\}$ of distinct pairs of integers such that $F(N(r'_n, s'_n)) = R(\sqrt{c})$ and $F(M(r'_n, s'_n)) = R(\sqrt{cd'_n})$ for some integers d'_n such that $(c, d'_n) = 1$.

There are five known pairs of integers (r, s) with $rs \neq 0$ and $s \neq -1$ such that $F(M(r, s))$ and $F(N(r, s))$ coincide. For $s \equiv 2 \pmod 4$ and for certain odd integers s the fields $F(M(r, s))$ and $F(N(r, s))$ cannot coincide for any integers r.

Finally, for any integer $r \neq 0$ (or $s \neq 0, -1$) there exist at most a finite number of integers s (or r) such that the two fields coincide.

Let $A = (a_{ij})$ be a matrix of order n with elements in the complex field. We say A is *normal* if and only if $\bar{A}'A = A\bar{A}'$ where $\bar{A}' = (\overline{a_{ji}})$. It is known that if A is normal, with characteristic roots λ_i, $i = 1, \cdots, n$, then[1] the characteristic roots of $A\bar{A}'$ are given by $\lambda_i \cdot \bar{\lambda}_i$, $i = 1, \cdots, n$. Conversely, if the characteristic roots of $A\bar{A}'$ can be written as $\lambda_i \cdot \bar{\lambda}_{\delta_i}$, $i = 1, \cdots, n$, where $\{\delta_1, \cdots, \delta_n\}$ is some permuta-

[1] This follows immediately from Theorem 1, [**1**].