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FRATTINI SUBGROUPS AND Φ-CENTRAL GROUPS

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Φ-central groups are introduced as a step in the direction of determining sufficiency conditions for a group to be the Frattini subgroup of some finite p-group and the related extension problem. The notion of Φ-centrality arises by uniting the concept of an E-group with the generalized central series of Kaloujnine. An E-group is defined as a finite group G such that $\Phi(N) \leqq \Phi(G)$ for each subgroup $N \leqq G$. If $\mathscr{H}$ is a group of automorphisms of a group N, N has an $\mathscr{H}$-central series $N = N_0 > N_1 > \cdots > N_r = 1$ if $x^{-1}x^{a} \in N_j$ for all $x \in N_{j-1}$, all $a \in \mathscr{H}$, x^a the image of x under the automorphism $a \in \mathscr{H}$, $j = 0, 1, \cdots, r-1$.

Denote the automorphism group induced on $\Phi(G)$ by transformation of elements of an E-group G by $\mathscr{H}$. Then $\Phi(\mathscr{H}) = \mathscr{I}(\Phi(G))$, $\mathscr{I}(\Phi(G))$ the inner automorphism group of $\Phi(G)$. Furthermore if G is nilpotent, then each subgroup $N \leqq \Phi(G)$, N invariant under $\mathscr{H}$, possess an $\mathscr{H}$-central series. A class of nilpotent groups N is defined as Φ-central provided that N possesses at least one nilpotent group of automorphisms $\mathscr{H} \neq 1$ such that $\Phi(\mathscr{H}) = \mathscr{I}(N)$ and N possesses an $\mathscr{H}$-central series. Several theorems develop results about Φ-central groups and the associated $\mathscr{H}$-central series analogous to those between nilpotent groups and their associated central series. Then it is shown that in a p-group, Φ-central with respect to a p-group of automorphism $\mathscr{H}$, a nonabelian subgroup invariant under $\mathscr{H}$ cannot have a cyclic center. The paper concludes with the permissible types of nonabelian groups of order p^4 that can be Φ-central with respect to a nontrivial group of p-automorphisms.

Only finite groups will be considered and the notation and the definitions will follow that of the standard references, e.g. [6]. Additionally needed definitions and results will be as follows: The group G is the *reduced partial product* (or reduced product) of its subgroups A and B if A is normal in $G = AB$ and B contains no subgroup K such that $G = AK$. For a reduced product, $A \cap B \leqq \Phi(B)$, (see [2]). If N is a normal subgroup of G contained in $\Phi(G)$, then $\Phi(G/N) \cong \Phi(G)/N$, (see [5]). An elementary group, i.e., an E-group having the identity for the Frattini subgroup, splits over each of its normal subgroups, (see [1]).

1. For a group G, $\Phi(G) = \Phi$, $G/\Phi = F$ is *Φ-free* i.e., $\Phi(F)$ is the identity. The elements of G by transformation of Φ induce auto-