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POINT-DETERMINING HOMOMORPHISMS ON MULTIPLICATIVE SEMI-GROUPS OF CONTINUOUS FUNCTIONS

R. S. DEZUR

Let X and Y be compact Hausdorff spaces, $C(X)$ and $C(Y)$ the algebras of real valued continuous functions on X and Y respectively with the usual sup norms. If T is an algebra homomorphism from $C(X)$ onto a dense subset of $C(Y)$ then by a theorem of Stone, T induces a homeomorphism μ from Y to X and it necessarily follows that $Tf(y) = 0$ if and only if $f(\mu(y)) = 0$.

In a more general setting, viewing $C(X)$ and $C(Y)$ as multiplicative semi-groups, let T be a semi-group homomorphism from $C(X)$ onto a dense point-separating set in $C(Y)$. No such map μ satisfying the above condition need exist. T is called point-determining in case for each y there is an x such that $Tf(y) = 0$ if and only if $f(x) = 0$. It is shown that such a homomorphism T induces a homeomorphism from Y into X in such a way that $Tf(y) = [\operatorname{sgn} f(x)]\,|f(x)|^{p(x)}$ for some continuous positive function p where x is related to y via the induced homeomorphism, that such a T is an algebra homomorphism followed by a semi-group automorphism, and that T is continuous.

Let X and Y be compact Hausdorff spaces, $C(X)$ and $C(Y)$ the algebras of all continuous real-valued functions on X and Y respectively with the usual sup norm. Let T be an algebra homomorphism of $C(X)$ onto a dense set in $C(Y)$. For each $y \in Y$ consider the mapping γ_y of $C(X)$ into the reals defined by

$$\gamma_y(f) = Tf(y) .$$

γ_y maps $C(X)$ onto the reals for if $Tf(x) = 0$ for all $f \in C(X)$ then the image of T is not dense. The kernel is, by algebra, a maximal ideal in $C(X)$. By a theorem of Stone [3, 80] there is a point $x \in X$ so that the kernel of γ_y is the set of all $f \in C(X)$ such that $f(x) = 0$, this point being uniquely determined.

Consider the map μ of Y into X which assigns to each $y \in Y$ the x as described above. If e and e_1 are the unit functions in $C(X)$ and $C(Y)$ respectively it is easy to see that $Te = e_1$ and that μ is one-to-one. Now for each $f \in C(X)$ consider the function $Tf(y)e - f = g$ in $C(X)$. Then $Tg(y) = 0$ so that $g(\mu(y)) = 0$ and hence $Tf(y) = f(\mu(y))$. We especially note that