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FIXED-POINT THEOREMS FOR FAMILIES OF CONTRACTION MAPPINGS

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Let X be a nonempty, bounded, closed and convex subset of a Banach space B. A mapping $f: X \to X$ is called a *contraction mapping* if $\|f(x) - f(y)\| \leqq \|x - y\|$ for all $x, y \in X$. Let $\mathfrak{F}$ be a nonempty commutative family of contraction mappings of X into itself. The following results are obtained.

(i) Suppose there is a compact subset M of X and a mapping $f_1 \in \mathfrak{F}$ such that for each $x \in X$ the closure of the set $\{f_1^n(x): n = 1, 2, \cdots\}$ contains a point of M (where f_1^n denotes the n^{th} iterate, under composition, of f_1). Then there is a point $x \in M$ such that $f(x) = x$ for each $f \in \mathfrak{F}$.

(ii) If X is weakly compact and the norm of B strictly convex, and if for each $f \in \mathfrak{F}$ the f-closure of X is nonempty, then there is a point $x \in X$ which is fixed under each $f \in \mathfrak{F}$. A third theorem, for finite families, is given where the hypotheses are in terms of weak compactness and a concept of Brodskii and Milman called normal structure.

Fixed-point theorems for families of continuous linear (or affine) transformations have been obtained by Kakutani [6], Markov [8], Day [2], and others. Recently De Marr [3] proved the following fixed-point theorem: If X is a nonempty, compact, convex subset of a Banach space B and if $\mathfrak{F}$ is a nonempty family of commuting contraction mappings of X into itself, then the family $\mathfrak{F}$ has a common fixed point in X. In Theorem 1 of this paper hypotheses of a type considered by Göhde in [5] are used to obtain a generalization of De Marr's result.

Throughout this paper we shall denote the *diameter* of a subset $A \subseteqq B$ by $\delta(A)$, i.e.,

$$\delta(A) = \sup\{\|x - y\|: x, y \in A\} .$$

THEOREM 1. *Let X be a nonempty, bounded, closed, convex subset of a Banach space B; let M be a compact subset of X. Let $\mathfrak{F}$ be a nonempty commutative family of contraction mappings of X into itself with the property that for some $f_1 \in \mathfrak{F}$ and for each $x \in X$ the closure of the set $\{f_1^n(x): n = 1, 2, \cdots\}$ contains a point of M. Then there is a point $x \in M$ such that $f(x) = x$ for each $f \in \mathfrak{F}$.*

Proof. Let K be a nonempty closed convex subset of X such that $f(K) \subseteqq K$ for each $f \in \mathfrak{F}$. Select a point $x \in K$. Since $f(K) \subseteqq K$, we have $\{f_1^n(x)\} \subseteqq K$. Hence it follows that