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ON CLOSED MAPPINGS, BICOMPACT SPACES, AND A PROBLEM OF P. ALEKSANDROV

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The purpose of this paper is to show, under very general circumstances, that if $f: X \to Y$ is a closed map, then $f^{-1}y$ must be bicompact for "most" $y \in Y$. Two theorems of this sort are obtained, one of which is then used to answer a question of P. Alexandroff on the effect of closed maps on countable-dimensional spaces.

If $f: X \to Y$ is a closed map, then it is known that, under suitable assumptions, $f^{-1}y$ has a bicompact boundary for all $y \in Y$ (see I. Vaĭnšteĭn [19], A. H. Stone [18], and K. Morita and S. Hanai [12]), and $f^{-1}y$ itself is bicompact for "most" $y \in Y$ (see K. Morita [11] and the author [4]). In §§1 and 2 of this paper, we prove two theorems of the latter kind, whose main feature is that they require minimal restrictions on X and no restriction at all (other than being T_1) on Y.

In §3, we give some applications of the results from §2. The most interesting among them is the following, which gives a complete answer to a question of P. Alexandroff, (Terminology is defined below).

THEOREM (3.1). *Let X be a countable-dimensional space with a countable net, and let $f: X \to Y$ be a closed mapping of X onto some uncountable-dimensional space Y. Then $Y_1 = \{y \in Y \mid \text{card}\,(f^{-1}y) \geqq \mathfrak{c}\}$ is uncountable-dimensional.*

Observe that Theorem 3.1 is new even in case X is compact metric. In that case, E. Skljarenko [15] has shown that Y_1 is not void, but his proof gives no further information about Y_1. Our proof is based on entirely different ideas.

Let me say here that I am very grateful to P. Alexandroff for valuable discussions about this question and to E. Michael for helping with the translation of this paper.

Notation and terminology. All spaces are completely regular (often is it sufficient to suppose T_1); all mappings are continuous, and all coverings are open. We call a family $\gamma = \{S\}$ of sets $S \subseteqq X$ a *net* in X, if, for every $x \in X$ and each open U containing x, there exists an $S \in \gamma$ with $x \in S \subseteqq U$ (see [3]). We write card A for the cardinality of A, and $\mathfrak{c}$ for the cardinality of the continuum. If γ is a family of subsets of a space X, and if $x \in X$, then γx denotes the union of all elements of γ containing x. As usual, we call a space *countable-dimensional* if it is a countable union of subspaces with dim $= 0$;