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OPERATORS WITH FINITE ASCENT AND DESCENT

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Let X be a Banach space and T a closed linear operator with range and domain in X. Let $\alpha(T)$ and $\delta(T)$ denote, respectively, the lengths of the chains of null spaces $N(T^K)$ and ranges $R(T^K)$ of the iterates of T. The Riesz region $\mathfrak{R}_T$ of an operator T is defined as the set of λ such that $\alpha(T-\lambda)$ and $\delta(T-\lambda)$ are finite. The Fredholm region $\mathfrak{F}_T$ is defined as the set of λ such that $n(T-\lambda)$ and $d(T-\lambda)$ are finite, $n(T)$ denoting the dimension of $N(T)$ and $d(T)$ the codimension of $R(T)$. It is shown that $\mathfrak{F}_T \cap \mathfrak{J}_T$ is an open set on the components of which $\alpha(T-\lambda)$ and $\delta(T-\lambda)$ are equal, when T is densely defined, with common value constant except at isolated points. Moreover, under certain other conditions, $\mathfrak{R}_T$ is shown to be open. Finally, some information about the nature of these conditions is obtained.

Let X denote an arbitrary Banach space and suppose that T is a linear operator with domain $D(T)$ and range $R(T)$ in X. We shall write $N(T)$ for the nullspace, $N(T) = \{x \in D(T): \ Tx = 0\}$.

Let $D(T^n) = \{x: \ x, Tx, \cdots, T^{n-1}x \in D(T)\}$ and define T^n on this domain by the equation $T^n x = T(T^{n-1}x)$ where n is any positive integer and $T^\circ = I$. It is a simple matter to verify that $\{N(T^k)\}$ forms an ascending sequence of subspaces. Suppose that for some k, $N(T^k) = N(T^{k+1})$; we shall then write $\alpha(T)$ for the smallest value of k for which this is true, and call the integer $\alpha(T)$, the *ascent* of T. If no such integer exists, we shall say that T has infinite ascent. In a similar way, $\{R(T^k)\}$ forms a descending sequence; the smallest integer for which $R(T^k) = R(T^{k+1})$ is called the *descent* of T and is denoted by $\delta(T)$. If no such integer exists, we shall say that T has infinite descent.

The quantities $\alpha(T)$ and $\delta(T)$ were first discussed by F. Riesz [4] in his original investigation of compact linear operators. A comprehensive treatment of the properties of $\alpha(T)$ and $\delta(T)$ can be found in [6] pp. 271–284. The purpose of the present work is the consideration of the functions $\alpha(\lambda I - T)$ and $\delta(\lambda I - T)$ for complex λ. When no confusion can arise, we shall write these quantities as $\alpha(\lambda)$ and $\delta(\lambda)$ respectively.

DEFINITION. Let $\mathfrak{R}_T$ denote the set $\{\lambda: \ \alpha(\lambda)$ and $\delta(\lambda)$ are finite$\}$. We shall refer to $\mathfrak{R}_T$ as the *Riesz region* of T.

If we write $n(\lambda)$ for the dimension of $N(\lambda I - T)$, i.e., the *nullity* of $\lambda I - T$ and $d(\lambda)$ for the codimension of $R(\lambda I - T)$, i.e.,