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ON THE Γ-RINGS OF NOBUSAWA

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N. Nobusawa recently introduced the notion of a Γ-ring, more general than a ring, and obtained analogues of the Wedderburn theorems for Γ-rings with minimum condition on left ideals. In this paper the notions of Γ-homomorphism, prime and (right) primary ideals, m-systems, and the radical of an ideal are extended to Γ-rings, where the defining conditions for a Γ-ring have been slightly weakened to permit defining residue class Γ-rings. The radical R of a Γ-ring M is shown to be an ideal of M, and the radical of M/R to be zero, by methods similar to those of McCoy. If M satisfies the maximum condition for ideals, the radical of a primary ideal is shown to be prime, and the ideal $Q \neq M$ is P-primary if and only if $P^n \subseteq Q$ for some n, and $AB \subseteq Q$, $A \not\subseteq P$ implies $B \subseteq Q$. Finally, in Γ-rings with maximum condition, if an ideal has a primary representation, then the usual uniqueness theorems are shown to hold by methods similar to those of Murdoch.

2. Preliminary definitions. If $M = \{a, b, c, \cdots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \cdots\}$ are additive abelian groups, and for all a, b, c in M and all α, β in Γ, the following conditions are satisfied

(0) $a\alpha b$ is an element of M,

(1) $(a + b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha + \beta)b = a\alpha b + a\beta b$, $a\alpha(b + c) = a\alpha b + a\alpha c$,

(2) $(a\alpha b)\beta c = a\alpha(b\beta c)$,

then M is called a Γ-ring. If these conditions are strengthened to

(0′) $a\alpha b$ is an element of M, $\alpha a\beta$ is an element of Γ,

(1′) same as (1),

(2′) $(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$,

(3′) $a\alpha b = 0$ for all a, b in M implies $\alpha = 0$,

we then have a Γ-ring in the sense of Nobusawa [3]. As indicated in [3], an example of a Γ-ring is obtained by letting X and Y be abelian groups, $M = \mathrm{Hom}(X, Y)$, $\Gamma = \mathrm{Hom}(Y, X)$, and $a\alpha b$ the usual composite map. (While Nobusawa does not explicitly require that M and Γ be abelian groups, it appears clear that this is intended.) We may note that it follows from (0)–(2) that $0\alpha b = a0b = a\alpha 0 = 0$ for all a and b in M and all α in Γ.

A subset A of the Γ-ring M is a right (left) *ideal* of M if A is an additive subgroup of M and $A\Gamma M = \{a\alpha c : a \in A, \alpha \in \Gamma, c \in M\}$ $(M\Gamma A)$ is contained in A. If A is both a left and a right ideal, then A is a two-sided ideal, or simply an ideal of M.