PACIFIC JOURNAL OF MATHEMATICS
Vol. 19, No. 2, 1966

FRACTIONAL POWERS OF OPERATORS

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A definition of fractional (or complex) powers A^α, $\alpha \in C$, is given for closed linear operators A in a Banach space X with the resolvent set containing the negative real ray $(-\infty, 0)$ and such that $\{\lambda(\lambda + A)^{-1}; 0 < \lambda < \infty\}$ is bounded; fundamental properties such as additivity ($A^\alpha A^\beta = A^{\alpha+\beta}$), coincidence with the iterations $A^\alpha = A^n$ for integers $\alpha = n$, and analytic dependence on α are discussed. Since the fractional powers A^α are generally unbounded in both of the cases Re $\alpha > 0$ and Re $\alpha < 0$, attention is paid to the domains $D(A^\alpha)$, which are related to the spaces D^σ and R^τ of $x \in X$ defined by the regularity of $(\lambda + A)^{-1}x$ at ∞ and 0. When $-A$ generates a bounded continuous semi-group or a bounded analytic semi-group, more detailed results are obtained.

The study of fractional powers of operators has a long history, which may go back to Abel's work on the tautochrone, the Riemann-Liouville integral, and its generalizations by M. Riesz. However, it is only recently that the general theory was developed. When A is the negative of the infinitesimal generator of a bounded semi-group of operators, Hille [6] and Phillips [16] showed that fractional powers could be treated in the framework of an operational calculus which they originated. This program was carried out thoroughly by Balakrishnan [1]. Later Balakrishnan [2] gave a new definition and extended his theory to a wider class of operators. About the same time two different definitions were introduced by Krasnosel'skii-Sobolevskii [13] and Kato [10]; further results were obtained by them, Yosida [22], Kato [11] and Watanabe [20]. These theories, with the exception of [11], as well as some classical results on the Riemann-Liouville integral (Hardy-Littlewood [5], Love-Young [14]), will be reconstructed from a unified point of view.

Our definition of fractional powers is essentially the same as Balakrishnan's second definition and if, in particular, Re $\alpha > 0$, they are identical. In order to see that this definition is a natural one, let us consider the case in which A is bounded and the resolvent set $\rho(A)$ contains the negative real axis $(-\infty, 0]$. The most natural definition of A^α for such an operator A is given by the Dunford integral

$$A^\alpha = \frac{1}{2\pi i} \int_\Gamma \zeta^\alpha (\zeta - A)^{-1} \, d\zeta \,, \tag{1.1}$$

where the path Γ encircles the spectrum $\sigma(A)$ counterclockwise avoiding