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# AN INEQUALITY FOR THE DENSITY OF THE SUM OF SETS OF VECTORS IN $n$-DIMENSIONAL SPACE

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**A Schnirelmann type density is defined for sets of "nonnegative" lattice points. If $A, B$ and $C = A + B$ are such sets with densities $\alpha, \beta$ and $\gamma$ respectively, then it is shown that $\gamma \geqq \beta/(1-\alpha)$ provided $\alpha + \beta < 1$.**

**1.** Let $n$ be a positive integer and let $Q$ be the set of all vectors $r = (\rho_1, \cdots, \rho_n)$ where each $\rho_i$ is a nonnegative integer and at least one $\rho_i$ is positive. We define a partial order relation $<$ on $Q$ where $r < s$ if and only if $\rho_i \leqq \sigma_i$ $(i = 1, 2, \cdots, n)$ with strict inequality holding for at least one index. Denote by $L(r)$ the set of all $x$ in $Q$ for which either $x < r$ or $x = r$.

A nonempty finite subset $F$ of $Q$ is called fundamental if, whenever $r \in F$, then $L(r) \subseteqq F$. For $A, X \subseteqq Q$ with $X$ finite, let $A(X)$ denote the number of vectors in $A \cap X$. Then the (Kvarda) density of $A$ is

$$\alpha = \text{glb}\,\frac{A(F)}{Q(F)}$$

where $F$ ranges over all fundamental subsets of $Q$.

Let $B \subseteqq Q$ and define $A + B = \{a, b, a + b \mid a \in A, b \in B\}$ where addition of vectors is done coordinatewise. Let $\beta$ and $\gamma$ be the densities of $B$ and $C = A + B$ respectively. Kvarda [1] has proved the inequaliy $\gamma = \alpha + \beta - \alpha\beta$ which for $n = 1$ was first proved by Landau and Schnirelmann. In this paper we prove $\gamma \geqq \beta/(1-\alpha)$ provided $\alpha + \beta < 1$. For $n = 1$, this has been proved by Schur [2].

## 2. Main results.

LEMMA 1. *Let $\bar{C}$ denote the complement of $C$ in $Q$ and suppose $\bar{C} \neq \Phi$. For a fundamental set $F$ let $F^*$ denote the set of maximal vectors of $F$ with respect to the partial ordering $<$. Then*

$$\gamma = \text{glb}\,\frac{C(F)}{Q(F)}$$

*where $F$ ranges over all fundamental sets with $F^* \subseteqq \bar{C}$.*

*Proof.* Let $\gamma'$ denote this glb. Clearly $\gamma \leqq \gamma'$. Let $G$ be any fundamental set. If $C(G) = Q(G)$ then $C(G)/Q(G) = 1 > \gamma'$. If $C(G) < Q(G)$ then $\bar{C} \cap G \neq \Phi$. In this case let $F$ be the union of all