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AN EMBEDDING THEOREM FOR FUNCTION SPACES

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Let G be an open set in E_n, and let $H_0^m(G)$ denote the Sobolev space obtained by completing $C_0^\infty(G)$ in the norm

$$||u||_m = \left\{\int_G \sum_{|\alpha|\leq m} |D^\alpha u(x)|^2 dx\right\}^{1/2} .$$

We show that the embedding maps $H_0^{m+1}(G) \subset H_0^m(G)$ are completely continuous if G is "narrow at infinity" and satisfies an additional regularity condition. This generalizes the classical case of bounded sets G.

As an application, the resolvent operator R_λ, associated with a uniformly strongly elliptic differential operator A with zero boundary conditions is completely continuous in $\mathscr{L}_2(G)$ provided G satisfies the same conditions. This generalizes a theorem of A. M. Molcanov.

Let G be an open set in Euclidean n-space E_n. Following standard usage, we denote by $C_0^\infty(G)$ the space of infinitely differentiable complex valued functions having compact support in G. Let $H_0^m(G)$ denote the Sobolev space obtained by completing $C_0^\infty(G)$ relative to the norm

$$||f||_m = \left\{\int_G \sum_{|\alpha|\leq m} |D^\alpha f(x)|^2 dx\right\}^{1/2} .$$

(See (3) below for notations.) It is an important and well-known result of functional analysis that each embedding

$$H_0^{m+1}(G) \subset H_0^m(G) , \qquad m = 0, 1, 2, \cdots$$

is completely continuous provided G is a bounded set. In this paper we show that this assumption can be relaxed; it turns out that a certain condition on G called "narrowness at infinity" (see Definition 2), which is obviously necessary, is also sufficient for complete continuity of the embeddings, provided G also satisfies a certain regularity condition. This result could be anticipated on the basis of theorems of F. Rellich [4] and A. M. Molcanov [3] concerning discreteness of the spectrum for the Laplace operator (with zero boundary conditions) on G.

DEFINITION 1. For an arbitrary open set $G \subset E_n$, with boundary ∂G, define

$$\rho(G) = \sup_{x\in G} \text{dist}\,(x, \partial G) . \tag{1}$$

Clearly $\rho(G)$ is the supremum of the radii of spheres inscribable in G.