PACIFIC JOURNAL OF MATHEMATICS
Vol. 19, No. 2, 1966

ON THE TRANSFORMATION OF INTEGRALS IN MEASURE SPACE

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One objective of this paper is to prove a formula for the transformation of integrals by means of a change of variable in purely measure—theoretic setting. The classical prototype of such formulas is the one in which the change of variable is effected by an (appropriately differentiable) one-to-one transformation from some subset of Euclidean n-space R^n onto some other subset of R^n; the jacobian of the transformation plays a key role here. For the present study the transformation which gives the change of variable is no longer assumed to be one-to-one but it is required to satisfy certain standard conditions relative to the measure spaces at hand.

Some of the results presented in this paper can be summarized informally as follows. Let T be a function from a nonempty set S onto a set X, let $\{S, \mathfrak{M}, \mu\}$ and $\{X, \mathfrak{N}, \nu\}$ be measure spaces, and let $\mathfrak{B}$ be a sub-σ-field of $\mathfrak{M}$. These entities are subjected to certain standard requirements. Within this basic setting is proved a formula which takes the form

$$\int_B (H \circ T) f \, d\mu = \int_{TB} H' W(., B) \, d\nu \; ; \tag{1}$$

in (1), H is some $\mathfrak{N}$-measurable function, B is a set in $\mathfrak{B}$, f is analogous to the jacobian, and $'W$ is a function having certain measure-theoretic properties. Indeed $'W(x, B)$ is intended to "count or weigh" the number of points in B mapped into x by T. In this paper certain theorems are proved which reveal in detail the relationship between f and $'W$.

Rado and Reichelderfer have developed in [5] a "general transformation formula" from which the classical formulas for the transformation of integrals can be derived. In [3] Craft extended this formula and Reichelderfer proved a transformation formula in 4.10 in [6] which extends Craft's result. Reichelderfer's formula applies not merely to Lebesgue integration in R^n (as the earlier formulas did) but rather is proved in a measure-theoretic (quasi-topological) setting. Formula (1) is also an extension of Craft's result. The theorem (see 3.1 below) in which (1) is proved neither implies nor is implied by the theorem in 4.10 in [6].

Thus the present paper is somewhat similar to [6] in purpose and spirit. For example, in this paper the concept of "weighing function" (the function $'W$ in (1) is a weighing function) is a generalization of the concept of multiplicity function discussed in [5]. The corresponding