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DIAGONABILITY OF IDEMPOTENT MATRICES

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A ring $\mathscr{R}$ (commutative with identity) with the property that every idempotent matrix over $\mathscr{R}$ is diagonable (i.e., similar to a diagonal matrix) will be called an *ID*-ring. We show that, in an *ID*-ring $\mathscr{R}$, if the elements $a_1, a_2, \cdots, a_n \in \mathscr{R}$ generate the unit ideal then the vector $[a_1, a_2, \cdots, a_n]$ can be completed to an invertible matrix over $\mathscr{R}$. We establish a canonical form (unique with respect to similarity) for the idempotent matrices over an *ID*-ring. We prove that if $\mathscr{N}$ is the ideal of nilpotents in $\mathscr{R}$ then $\mathscr{R}$ is an *ID*-ring if and only if $\mathscr{R}/\mathscr{N}$ is an *ID*-ring. The following are then shown to be *ID*-rings: elementary divisor rings, a restricted class of Hermite rings, π-regular rings, quasi-semi-local rings, polynomial rings in one variable over a principal ideal ring (zero divisors permitted), and polynomial rings in two variables over a π-regular ring with finitely many idempotents.

In this paper, $\mathscr{R}$ will denote a commutative ring with identity, and $\mathscr{R}_n$ will denote the set of $n \times n$ matrices over $\mathscr{R}$. If $A, B \in \mathscr{R}_n$, then $A \cong B$ will mean that A is similar to B. We remark that if $\mathscr{R}$ is an *ID*-ring then every finitely generated projective R-module is the finite direct sum of cyclic modules, and that $\mathscr{R}$ is a directly indecomposable *ID*-ring if and only if every finitely generated projective $\mathscr{R}$-module is free. Most of the literature on this subject has been concerned with showing that a given ring $\mathscr{R}$ has the property that every finitely generated projective $\mathscr{R}$-module is free. This necessarily imposes the condition that $\mathscr{R}$ be indecomposable. In this paper, no such restriction is made.

2. Properties of *ID*-rings.

DEFINITION 1. $\mathscr{R}$ is said to be an *ID*-ring provided that for every $A = A^2 \in \mathscr{R}_n$, $n = 1, 2, \cdots$, there exists an invertible matrix $P \in \mathscr{R}_n$ such that PAP^{-1} is a diagonal matrix.

DEFINITION 2. The row vector $[a_1, a_2, \cdots, a_n]$ with components in $\mathscr{R}$ is said to be a basal provided that it can be completed to an invertible matrix over $\mathscr{R}$.

DEFINITION 3. The row vector X is said to be a characteristic vector of $A \in \mathscr{R}_n$ corresponding to $r \in \mathscr{R}$ provided (1) X is a basal vector and (2) $XA = rX$.