PACIFIC JOURNAL OF MATHEMATICS
Vol. 19, No. 3, 1966

MODULAR PAIRS IN ORTHOMODULAR LATTICES

ERIK A. SCHREINER

Call an orthomodular lattice L M-symmetric if $M(e,f)$ implies $M(f,e)$ for all $e,f \in L$ and O-symmetric if $M(e,f)$ implies $M(f',e')$. To check for these properties it is sufficient to consider only those modular pairs in which the two elements are complements. Every O-symmetric lattice is M-symmetric. In an atomic orthomodular lattice, M-symmetry is equivalent to the atomic exchange property.

The orthomodular lattice $L(H)$ of closed subspaces of a Hilbert space H satisfies both symmetries but apparently for separate reasons. G. W. Mackey has shown [4, Theorem III-6] that two closed subspaces of H form a dual modular pair in $L(H)$ if and only if their vector sum is a closed subspace. Thus the natural symmetry, M-symmetry, depends on the topology of H. O-symmetry arises in $L(H)$ as a consequence of properties of bounded linear operators on H with closed ranges. It is the purpose of this paper to investigate these symmetries in arbitrary orthomodular lattices. Recent results of A. Ramsay [8] and M. D. MacLaren [7] have shown M-symmetry to be of importance in the study of locally finite dimension lattices. The Baer $*$-semigroup coordinatization theory for orthomodular lattices, developed by D. J. Foulis [2], enables us to adapt the idea of an operator with a closed range and conveniently and naturally introduce O-symmetry into arbitrary orthomodular lattices.

Confining our attention to orthomodular lattices, we first establish certain general properties of modular pairs. Using these results we develop characterizations of M-symmetric lattices and O-symmetric lattices in Theorem 7 and Theorem 8. That O-symmetry implies M-symmetry is established in Theorem 9. In § 4, atomic orthomodular lattices satisfying the atomic exchange property introduced by MacLane [5] are considered. Using specializations of the approach and procedure used by Ramsay in [8] it is shown that M-symmetry is equivalent to the atomic exchange property.

The author would like to express his appreciation to Professor D. J. Foulis for the many helpful suggestions he has made during the writing of this paper.

2. Modular pairs. In order to establish certain properties of modular pairs in orthomodular lattices, we shall make use of the Baer $*$-semigroup approach as defined and developed in [1, 2, 3]. We shall restrict our resume of definitions and results to a minimum of