ON A PROBLEM OF O. TAUSSKY

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Recently, O. Taussky raised the following question. Given a nonnegative $n \times n$ matrix $A = (a_{i,j})$, let $\hat{Q}_A$ be the set of all $n \times n$ complex matrices defined by

$$\hat{Q}_A = \{ B = (b_{i,j}) \mid |b_{i,j}| = a_{i,j} \text{ for all } 1 \leq i, j \leq n \}.$$  

Then, defining the spectrum $S(\hat{Q})$ of an arbitrary set $\hat{Q}$ of $n \times n$ matrices $B$ as

$$S(\hat{Q}) \equiv \{ \lambda \mid \det(\lambda I - B) = 0 \text{ for some } B \in \hat{Q} \},$$

what can be said in particular about $S(\hat{Q}_A)$? It is not difficult to see that $S(\hat{Q}_A)$ consists of possibly one disk and a series of annular regions concentric about the origin, but our main result is a precise characterization of $S(\hat{Q}_A)$ in terms of the minimal Gerschgorin sets for $A$.

**Introduction.** We shall distinguish between two cases. If there is a diagonal matrix $D = \text{diag}(x, \ldots, x)$ with $x \geq 0$ and $x \neq 0$ such that $AD$ is diagonally dominant, then $A$ is called essentially diagonally dominant. In this case, the set $S(\hat{Q}_A)$ is just the minimal Gerschgorin set $G(\hat{Q}_A)$ of $[6]$, rotated about the origin (Theorem 1 and Corollary 2). Determining $S(\hat{Q}_A)$ in this case is quite easy, since it suffices to determine those points of the boundary of $G(\hat{Q}_A)$ which lie on the positive real axis (Theorem 2). This is discussed in §2.

In the general case when $A$ is not essentially diagonally dominant, we must use permutations and intersections (Theorem 3) to fully describe $S(\hat{Q}_A)$, in the spirit of [3]. These results are described in §3. Also in this section is a generalization (Theorems 3 and 4) of a recent interesting result by Camion and Hoffman [1]. Our proof of this generalization differs from that of [1].

Finally, in §4 we give several examples to illustrate the various possibilities for $S(\hat{Q}_A)$.

Before leaving this section, we point out that the question posed by O. Taussky [5, p. 129] has an immediate answer in terms of the results of [3]. In [3], the authors completely characterized the spectrum $S(\hat{Q}_C)$ of a related set $\hat{Q}_C$ of matrices, where $C = (c_{i,j})$ was an arbitrary $n \times n$ complex matrix and

$$\hat{Q}_C \equiv \{ B = (b_{i,j}) \mid |b_{i,j}| = |c_{i,j}| \text{ and } b_{i,j} = c_{i,j} \text{ for all } 1 \leq i, j \leq n \}.$$  

Clearly, $\hat{Q}_A \subset \hat{Q}_C$. On the other hand, if $D(\theta)$ represents an $n \times n$ diagonal matrix all of whose diagonal entries have modulus unity: