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ON A PROBLEM OF O. TAUSSKY

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Recently, O. Taussky raised the following question. Given a nonnegative $n \times n$ matrix $A = (a_{i,j})$, let $\mathring{\Omega}_A$ be the set of all $n \times n$ complex matrices defined by

$$(1.1) \quad \mathring{\Omega}_A \equiv \{B = (b_{i,j}) \mid |b_{i,j}| = a_{i,j} \text{ for all } 1 \leqq i, j \leqq n\} .$$

Then, defining the spectrum $S(\mathfrak{M})$ of an arbitrary set $\mathfrak{M}$ of $n \times n$ matrices B as

$$(1.2) \quad S(\mathfrak{M}) \equiv \{\sigma \mid \det(\sigma I - B) = 0 \text{ for some } B \in \mathfrak{M}\} ,$$

what can be said in particular about $S(\mathring{\Omega}_A)$? It is not difficult to see that $S(\mathring{\Omega}_A)$ consists of possibly one disk and a series of annular regions concentric about the origin, but our main result is a precise characterization of $S(\mathring{\Omega}_A)$ in terms of the minimal Gerschgorin sets for A.

Introduction. We shall distinguish between two cases. If there is a diagonal matrix $D = \text{diag}(x_1, \cdots, x_n)$ with $\boldsymbol{x} \geqq \boldsymbol{0}$ and $\boldsymbol{x} \neq \boldsymbol{0}$ such that AD is diagonally dominant, then A is called essentially diagonally dominant. In this case, the set $S(\mathring{\Omega}_A)$ is just the minimal Gerschgorin set $G(\Omega_A)$ of [6], *rotated* about the origin (Theorem 1 and Corollary 2). Determining $S(\mathring{\Omega}_A)$ in this case is quite easy, since it suffices to determine those points of the boundary of $G(\Omega_A)$ which lie on the positive real axis (Theorem 2). This is discussed in § 2.

In the general case when A is not essentially diagonally dominant, we must use permutations and intersections (Theorem 3) to fully describe $S(\mathring{\Omega}_A)$, in the spirit of [3]. These results are described in § 3. Also in this section is a generalization (Theorems 3 and 4) of a recent interesting result by Camion and Hoffman [1]. Our proof of this generalization differs from that of [1].

Finally, in § 4 we give several examples to illustrate the various possibilities for $S(\mathring{\Omega}_A)$.

Before leaving this section, we point out that the question posed by O. Taussky [5, p. 129] has an immediate answer in terms of the results of [3]. In [3], the authors completely characterized the spectrum $S(\Omega_C)$ of a related set Ω_C of matrices, where $C = (c_{i,j})$ was an arbitrary $n \times n$ complex matrix and

$$(1.3) \quad \Omega_C \equiv \{B = (b_{i,j}) \mid |b_{i,j}| = |c_{i,j}| \text{ and } b_{i,i} = c_{i,i} \text{ for all } 1 \leqq i, j \leqq n\} .$$

Clearly, $\Omega_A \subset \mathring{\Omega}_A$. On the other hand, if $D(\boldsymbol{\theta})$ represents an $n \times n$ diagonal matrix all of whose diagonal entries have modulus unity: