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CLOSED AND IMAGE-CLOSED RELATIONS

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If X and Y are topological spaces, a relation $T \subseteq X \times Y$ is upper semi-continuous at the point x of the domain $D(T)$ of T if for each neighborhood V of $T(x)$, there is a neighborhood U of x such that $T(U) \subseteq V$. Results so far published about such relations usually require that they be closed (as subsets of the product space) or image-closed ($T(x)$ is closed in Y for each $x \in X$). Given any relation T, it seems natural to consider the associated relations T' and $\bar{T}$, where T' is defined by $T'(x) = \overline{T(x)}$ and $\bar{T}$ is the closure of T in the product space. In particular, it is pertinent to ask under what conditions the upper semi-continuity of T implies that of T' or $\bar{T}$, or that $T' = \bar{T}$. As might be expected, the answers to these questions take the form of restrictions on Y, and, indeed, serve to characterize regularity, normality, and compactness.

Other relation-theoretic characterizations have been given previously. In [6], Engelking characterizes regularity and compactness (in two ways), and in [10], Michael characterizes normality, collectionwise normality, perfect normality, and paracompactness. Ceder [1] characterizes $\mathfrak{m}$-compactness.

Terminology in this paper will follow Kelley [9]; in particular, regular and normal spaces need not be T_1. The following well known fact will be used: T is upper semi-continuous (hereinafter abbreviated usc) on $D(T)$ if and only if the inverse under T of each closed subset of Y is closed in $D(T)$. A relation $T \subseteq X \times Y$ will be said to be *on* X *into* Y if and only if $D(T) = X$.

Statement of results. These are arranged so that for $n = 1, 2, 3, 4$, result ($2n$) is in the nature of a converse of result ($2n - 1$), thus yielding the promised characterizations of regularity, normality, and various types of compactness.

(1) *If Y is regular and $T \subseteq X \times Y$ is usc at $x \in D(T)$, then* $T'(x) = \bar{T}(x)$.

Regularity of Y does not imply the upper semi-continuity of T' or $\bar{T}$ for usc $T \subset X \times Y$ (see (6a) and (6b) below).

The statement of the next result, a converse of (1), and of several others will be expedited by a definition: Let Δ be a directed set and $p \notin \Delta$. Define a topology for $X = \Delta \cup \{p\}$ by letting each point of Δ be isolated and taking as a base at p all sets of the form $S \cup \{p\}$ where