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SOME METRICAL THEOREMS IN NUMBER THEORY

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In this paper some metrical theorems on Diophantine approximation, continued fractions and θ-adic expansions are proved.

In the first part some of the common properties of the following transformations from the unit interval onto itself are investigated. Denote by $\{\alpha\}$ the fractional part of x,

A. $T : \alpha \to \{a\alpha\}$ $\quad a > 1$ integer

which describes the expansion of α in the scale a

B. $T : \alpha \to \left\{\frac{1}{\alpha}\right\}$

which describes the continued fractions

C. $T : \alpha \to \{\theta\alpha\}$ $\quad \theta > 1$ noninteger

which describes the expansion of α as a θ-adic fraction.

The main theorem of the first part (Theorem 2) gives an estimate of the number of solutions of the system of inequalities

$$T^k\alpha \in I_k \qquad 1 \leqq k \leqq n$$

where n is an integer, T is any of these three transformations and (I_k) is an arbitrary sequence of intervals contained in the unit interval.

It generalises and refines well known theorems on the distribution function of the sequence $(T^k\alpha)$. Theorem 2 follows from a very general theorem—a quantitative Borel-Cantelli Lemma.

It is also shown that T is strongly mixing (Theorem 1). The second part of the paper deals with the metric theory of continued fractions. Theorems of LeVeque and Bernstein are refined.

1. Frequently a real number α is represented in one of the following ways:

A. in the scale a, where $a > 1$ is an integer,

B. as a continued fraction,

C. as a θ-adic fraction, where $\theta > 1$ is a noninteger.

Let us recall some of the properties of these representations:

A. If $a > 1$ denotes an integer then every $\alpha \in [0, 1)$ can be written as

$$\alpha = \sum_{k=1}^{\infty} \frac{c_k}{a^k} = \sum_{k=1}^{n} \frac{c_k}{a^k} + \frac{y_{n+1}}{a^n}$$