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ON CHARACTERIZING THE GAMMA AND THE NORMAL DISTRIBUTION

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We will characterize the gamma distribution by the nature of the joint distribution of the two quotients X_1/X_3, X_2/X_3 for three identically gamma distributed random variables.

It is well known that if two independent identically distributed random variables X_1, X_2 have the gamma distribution given by the density

$$f(x) = \begin{cases} 0 & \text{for } x \leqq 0 \\ \dfrac{a^p}{\Gamma(p)} x^{p-1} e^{-ax} & \text{for } x > 0 \end{cases} \qquad \begin{pmatrix} a > 0 \\ p > 0 \end{pmatrix} \tag{1}$$

then their quotient

$$Y = X_1/X_2 \tag{2}$$

has the beta distribution of the second kind given by the density

$$g(y) = \begin{cases} 0 & \text{for } y \leqq 0 \\ \dfrac{1}{B(p, p)} \cdot \dfrac{y^{p-1}}{(1+y)^{2p}} & \text{for } y > 0 . \end{cases} \tag{3}$$

However, this property does not characterize the gamma distribution uniquely. There exist pairs of independent positive identically distributed random variables X_1, X_2 whose common distribution function $F(x)$ differs from the one given by the density (1), but where the quotients (2) are distributed according to the density (3). Some such distribution functions $F(x)$ are given by the following densities

$$f_1(x) = \begin{cases} 0 & \text{for } x \leqq 0 \\ \dfrac{a^p}{\Gamma(p)} x^{-p-1} e^{-a/x} & \text{for } x > 0 \end{cases}$$

$$f_2(x) = \begin{cases} 0 & \text{for } x \leqq 0 \\ \dfrac{2\Gamma\left(\dfrac{2p+1}{2}\right)}{\Gamma\left(\dfrac{p}{2}\right)\Gamma\left(\dfrac{p+1}{2}\right)} \cdot \dfrac{x^{p-1}}{(1+x^2)^{p+1/2}} & \text{for } x > 0 \end{cases} \tag{4}$$