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REPRESENTATIONS OF DIRECT PRODUCTS OF FINITE GROUPS

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Let G be a finite group and K an arbitrary field. We denote by $K(G)$ the group algebra of G over K. Let G be the direct product of finite groups G_1 and G_2, $G = G_1 \times G_2$, and let M_i be an irreducible $K(G_i)$-module, $i = 1, 2$. In this paper we study the structure of M_1, M_2, the outer tensor product of M_1 and M_2.

While M_1, M_2 is not necessarily an irreducible $K(G)$-module, we prove below that it is completely reducible and give criteria for it to be irreducible. These results are applied to the question of whether the tensor product of division algebras of a type arising from group representation theory is a division algebra.

We call a division algebra D over K K-derivable if $D \cong \mathrm{Hom}_{K(G)}(M, M)$ for some finite group G and irreducible $K(G)$-module M. If $B(K)$ is the Brauer group of K, the set $B_0(K)$ of classes of central simple K-algebras having division algebra components which are K-derivable forms a subgroup of $B(K)$. We show also that $B_0(K)$ has infinite index in $B(K)$ if K is an algebraic number field which is not an abelian extension of the rationals.

All $K(G)$-modules considered are assumed to be unitary finite dimensional left $K(G)$-modules. If M_i is a $K(G_i)$-module, $i = 1, 2$, the outer tensor product $M_1 \# M_2$ of M_1 and M_2 is the $K(G)$-module whose underlying space is $M_1 \otimes_K M_2$ and where $(g_1, g_2) \in G$ acts on $M_1 \otimes_K M_2$ by

$$(g_1, g_2) \sum m_i \otimes m_i' = \sum g_1 m_i \otimes g_2 m_i', \; m_i \in M_1, \; m_i' \in M_2, \; g_j \in G_j, \; j = 1, 2 .$$

It will be necessary to refer to the theory of the Schur index of absolutely irreducible representations of finite groups. In § 1 we present a treatment of this theory where the relevant theorems are proved for arbitrary fields. This treatment is included in the author's doctoral dissertation supervised by Professor Charles W. Curtis at the University of Oregon. During the preparation of this paper the author held a National Science Foundation Graduate Fellowship.

1. The Schur index. The method used in [3, § 70] to prove the relevant theorems about the Schur index for fields of characteristic zero does not seem to generalize to arbitrary fields. In that treatment attention is focused on the enveloping algebra of the representations rather than on the representations themselves. We work directly with modules