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# HOLONOMY GROUPS OF INDEFINITE METRICS

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**This paper studies the holonomy group of a riemannian manifold whose metric is allowed to have arbitrary signature; it is meant to supplement the works of Borel, Lichnerowicz and Berger on riemannian manifolds with positive definite metric. We first show that each such holonomy group can be decomposed into the direct product of a finite number of weakly irreducible subgroups of the pseudo-orthogonal group. Those weakly irreducible subgroups which are not irreducible (in the usual sense) we call $S$-$W$ irreducible. So our investigation is reduced to that of these $S$-$W$ irreducible holonomy groups. We actually construct a large class of symmetric spaces with $S$-$W$ irreducible holonomy groups and for the nonsymmetric case, we give an indication of their abundant existence. On the other hand, not every $S$-$W$ irreducible group can be realized as a holonomy group; this fact is shown by an explicit example. We then study the closedness question of $S$-$W$ irreducible subgroups in general, and of holonomy groups in particular. It turns out that *algebraic* holonomy groups (and hence $S$-$W$ irreducible subgroups in general) need not be closed in $Gl_n$ but that holonomy groups of symmetric riemannian manifolds of any signature are necessarily closed. Sufficient conditions are also given in order that an $S$-$W$ irreducible subgroup be closed. Finally, we produce various counterexamples to show that many facts known to hold in the positive definite case fail when the metric is allowed to be indefinite.**

In two exhaustive works [2], [3], Berger has given a complete classification of possible candidates for *irreducible* holonomy groups of riemannian manifolds. (In this paper, "holonomy groups" is synonymous with "the identity component of the homogeneous holonomy group" and riemannian manifolds carry metrics of arbitrary signatures. For all relevant conventions and definitions, see Section 2). Since Borel and Lichnerowicz have shown [4] that for the positive definite case, every holonomy group is the direct product of irreducible subgroups of the orthogonal group, the consideration of irreducible holonomy groups alone is sufficient for that case. On the other hand, if the metric is indefinite, the situation becomes different. Defining a subgroup of the pseudo-orthogonal group $PO(V)$ to be weakly irreducible if and only if it leaves invariant only proper degenerate subspaces of $V$, we have the following simple but basic result.

THEOREM 1. *The holonomy group of a riemannian manifold is*