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THE DEDEKIND COMPLETION OF $C(\mathscr{X})$

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The question to which this study addresses itself is the following: given a completely regular space $\mathscr{X}$, is the Dedekind completion of $C(\mathscr{X})$ isomorphic to $C(\mathscr{Y})$ for some space $\mathscr{Y}$? Here, $C(\mathscr{X})$ denotes the ring of continuous real-valued functions on $\mathscr{X}$ under pointwise order. Affirmative answers were provided by Dilworth for the class of compact spaces in 1950 and by Weinberg for the class of countably paracompact and normal spaces in 1960. It remained an open question whether there were any spaces for which a negative answer held. In this paper, we provide a necessary and sufficient condition that the Dedekind completion of $C(\mathscr{X})$, for $\mathscr{X}$ a realcompact space, be isomorphic to $C(\mathscr{Y})$ for some $\mathscr{Y}$. Using this, we are able to provide an example of a space $\mathscr{X}$ for which the Dedekind completion of $C(\mathscr{X})$ is not isomorphic to $C(\mathscr{Y})$ for any space $\mathscr{Y}$.

Specifically, we define and characterize a class of spaces which we call *weak cb-spaces*: those spaces $\mathscr{X}$ with the property that every locally bounded, lower semicontinuous function on $\mathscr{X}$ is bounded above by a continuous function. We then prove that for an arbitrary (completely regular) space $\mathscr{X}$, the Dedekind completion of $C(\mathscr{X})$ is isomorphic to some $C(\mathscr{Y})$ if and only if $\nu\mathscr{X}$ (the Hewitt realcompactification of $\mathscr{X}$) is a weak *cb*-space. The sufficiency of this condition actually generalizes Weinberg's result, as is shown by examples; the necessity provides the negative result referred to above.

The preliminary investigation of the Dedekind completion is done in Section 1, in the setting of an arbitrary Φ-algebra. In Section 2, we study the connection between the lattice of normal upper semicontinuous functions on a completely regular space $\mathscr{X}$ and the minimal projective extension of $\mathscr{X}$. This leads to the observation, in Section 4, that for a weak *cb*-space $\mathscr{X}$ the Dedekind completion of $C(\mathscr{X})$ is isomorphic to $C(\mathscr{Y})$, where $\mathscr{Y}$ is the minimal projective extension of $\mathscr{X}$. Weak *cb*-spaces are studied in Section 3, and Section 4 contains our main result.

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The Dedekind completion of an arbitrary Φ-algebra. A *Φ-algebra* is a real archimedean lattice-ordered algebra with identity