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CONTINUITY OF TRANSFORMATIONS WHICH LEAVE INVARIANT CERTAIN TRANSLATION INVARIANT SUBSPACES

B. E. JOHNSON

It is shown that a linear operator $T: L^2(X) \to L^2(X)$ (X a locally compact group), with the property that $TE \subset E$ for each norm closed right translation invariant subspace E of $L^2(X)$, is necessarily continuous. In § 5 the author shows that this is also true for $L^1(X)$ when X contains an element a which does not lie in any compact subgroup. An example is constructed to show that, in $l^\infty(-\infty, +\infty)$, T can be discontinuous and still leave invariant each $\sigma(l^\infty, l^1)$ closed translation invariant subspace of l^∞. If however $T: l^\infty(-\infty, +\infty) \to l^\infty(-\infty, +\infty)$ leaves invariant all norm closed translation invariant subspaces, then T must be continuous.

We shall use the notation of [3] without further explanation. Sections 3, 4 and 5 overlap with some of the results in 1.4 of Edwards' paper [3] where he shows *inter alia* that T is automatically continuous in the L^2 case if X is compact and in the L^1 case if X is finitely representable. The work in § 6 answers the conjecture in 1.7 of [3].

2. **Three basic lemmas.** For the first lemma let $\mathfrak{X}$ be a normal topological space and $\mathscr{A}$ a set of complex valued functions on $\mathfrak{X}$, closed with respect to pointwise multiplication, containing the constant function 1 and such that if $f \in \mathscr{A}$ then so is $1 - f$. We assume that $\mathscr{A}$ is normal in the sense that if F_0 and F_1 are closed disjoint subsets of $\mathfrak{X}$ then there is a function $f \in \mathscr{A}$ with $f(F_0) = \{0\}, f(F_1) = \{1\}$. We suppose that there is a mapping $f \to S_f$ of $\mathscr{A}$ into $\mathscr{B}(E)$, where E is some Banach space, such that $S_{fg} = S_f S_g$, $S_1 = I$, $S_{1-f} = I - S_f$. We denote the null space of S_f by N_f and suppose that T is a linear map of E into E which leaves invariant all the subspaces $N_f (f \in \mathscr{A})$. For each $f \in \mathscr{A}$ the composition of T with the quotient map $E \to E/N_f$ gives a map $T_f: E \to E/N_f$.

DEFINITION 2.1. $\lambda \in \mathfrak{X}$ is called a *discontinuity value* of T if T_f is discontinuous whenever $f \in \mathscr{A}$ and there exists a neighbourhood N of λ such that $f(\lambda') \neq 0$ for all $\lambda' \in N$.

LEMMA 2.2. *T has only a finite number of discontinuity values.*

Proof. Suppose T has a infinity of discontinuity values. Then