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INEQUALITIES FOR FUNCTIONS REGULAR AND BOUNDED IN A CIRCLE

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This paper is concerned with functions $w = f(z)$ regular and satisfying the inequality $|f(z)| < 1$ in $|z| < 1$. This class of functions will be denoted E.

We determine conditions on z_1, z_2, z_3 and w_1, w_2, w_3 for

$$w_k = f(z_k)\,(k = 1, 2, 3)$$

to be possible with an $f(z)$ of E. In particular to map the vertices of the equilateral triangle $z_k = re^{2k\pi i/3}$ into the vertices of another taken in the opposite direction $w_k = \rho e^{-2k\pi i/3}$ we must have $\rho \leqq r^2$. The extremal function associated with this problem is $w = z^2$. It seems convenient to discuss the fixed point if any of the mapping of $|z| < 1$ into $|w| < 1$. We include a simple proof of the theorem of Denjoy and Wolff that if no such fixed point exists then there is a unique distinguished fixed point on $|z| = 1$. We give several results restricting the position of the interior or distinguished boundary fixed point in terms of the location of a zero of $f(z)$ or the value $f(0)$.

The theorem of Pick asserts that if $f(z)$ is in E then $D(f(z_1), f(z_2)) \leqq D(z_1, z_2)$ where the nonEuclidean distance

$$D(z_1, z_2) = \frac{1}{2} \log \frac{1 + d(z_1, z_2)}{1 - d(z_1, z_2)} \text{ with } d(z_1, z_2) = \left| \frac{z_1 - z_2}{1 - \bar{z}_2 z_1} \right| .$$

Equality holds if and only if f sets up a Möbius transformation. It follows from Pick's theorem that there can be at most one fixed point of $w = f(z)$ in $|z| < 1$ unless $f(z) \equiv z$. It is usually sufficient when f has an interior fixed point at $z = \alpha (\neq 0)$ to suppose $0 < \alpha < 1$.

Our first four theorems give information about the relative positions of zeros of f, an interior fixed point, and the value $f(0)$. We exclude the case where $f(z) \equiv z$.

THEOREM 1. *Let $f \in E$ and $f(0) \neq 0$. Then f has no zeros in $|z| < |f(0)|$; and has a zero on $|z| = |f(0)|$ if and only if f determines a Möbius transformation.*

Proof. The image of $|z| \leqq |f(0)|$, which we denote by C under the transformation $w = (z + f(0))/(1 + \overline{f(0)}z)$ is a circular disc C' having nonEuclidean center $f(0)$ with boundary passing through the origin. The function $w = f(z)$ takes the closed disc C inside C' in the case f