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FRACTIONAL POWERS OF OPERATORS, II INTERPOLATION SPACES

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This is a continuation of an earlier paper "Fractional Powers of Operators" published in this Journal concerning fractional powers A^α, $\alpha \in C$, of closed linear operators A in Banach spaces X such that the resolvent $(\lambda + A)^{-1}$ exists for all $\lambda > 0$ and $\lambda(\lambda + A)^{-1}$ is uniformly bounded. Various integral representations of fractional powers and relationship between fractional powers and interpolation spaces, due to Lions and others, of X and domain $D(A^\alpha)$ are investigated.

In § 1 we define the space $D_p^\sigma(A)$, $0 < \sigma < \infty$, $1 \leqq p \leqq \infty$ or $p = \infty -$, as the set of all $x \in X$ such that

$$\lambda^\sigma(A(\lambda + A)^{-1})^m x \in L^p(X)\ ,$$

where m is an integer greater than σ and $L^p(X)$ is the L^p space of X-valued functions with respect to the measure $d\lambda/\lambda$ over $(0, \infty)$.

In § 2 we give a new definition of fractional power A^α for Re $\alpha > 0$ and prove the coincidence with the definition given in [2]. Convexity of $\| A^\alpha x \|$ is shown to be an immediate consequence of the definition. The main result of the section is Theorem 2.6 which says that if $0 < \operatorname{Re} \alpha < \sigma$, $x \in D_p^\sigma$ is equivalent to $A^\alpha x \in D_p^{\sigma - \operatorname{Re} \alpha}$. In particular, we have $D_1^{\operatorname{Re} \alpha} \subset D(A^\alpha) \subset D_{\infty -}^{\operatorname{Re} \alpha}$. For the application of fractional powers it is important to know whether the domain $D(A^\alpha)$ coincides with $D_p^{\operatorname{Re} \alpha}$ for some p. We see, as a consequence of Theorem 2.6, that if we have $D(A^\alpha) = D_p^{\operatorname{Re} \alpha}$ for an α, it holds for all Re $\alpha > 0$. An example and a counterexample are given. At the end of the section we prove an integral representation of fractional powers.

Section 3 is devoted to the proof of the coincidence of D_p^σ with the interpolation space $S(p, \sigma/m, X; p, \sigma/m - 1, D(A^m))$ due to Lions-Peetre [4]. We also give a direct proof of the fact that $D_p^\sigma(A^\alpha) = D_p^{\alpha\sigma}(A)$.

In § 4 we discuss the case in which $-A$ is the infinitesimal generator of a bounded strongly continuous semi-group T_t. A new space $C_{p,m}^\sigma$ is introduced in terms of $T_t x$ and its coincidence with D_p^σ is shown. Since $C_{\infty,m}^\sigma$, $\sigma \neq$ integer, coincides with C^σ of [2], this solves a question of [2] whether $C^\sigma = D^\sigma$ or not affirmatively. The coincidence of $C_{p,m}^\sigma$ with $S(p, \sigma/m, X; p, \sigma/m - 1, D(A^m))$ has been shown by Lions-Peetre [4]. Further, another integral representation of fractional powers is obtained.