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A NOTE ON DAVID HARRISON'S THEORY OF PREPRIMES

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A Stone ring is a partially ordered ring K with unit element 1 satisfying (1) 1 is positive; (2) for every x in K there exists a natural number n such that $n \cdot 1 - x$ belongs to K; and (3) if $1 + nx$ is positive for all natural numbers n then x is positive. Our first theorem: Every Stone ring is order-isomorphic with a subring of the ring of all continuous real functions on some compact Hausdorff space, with the usual partial order. A corollary is a theorem first proved by Harrison: Let K be a partially ordered ring satisfying conditions (1) and (2), and suppose the positive cone of K is maximal in the family of all subsets of K which exclude -1 and are closed under addition and multiplication. Then K is order-isomorphic with a subring of the reals.

The present paper is inspired by David Harrison's recently begun program of arithmetical ring theory where the basic objects are primes and preprimes; the positive cones of a ring are example of preprimes.

Throughout the paper, K will be a ring with unit element 1, and N will denote the set of positive integers. A *preprime* P in K is a nonempty subset of K excluding -1 and closed under addition and multiplication. A *prime* in K is a preprime maximal relative to set inclusion. A preprime P is *infinite* provided it contains both zero and 1, and is *conic* if $P \cap (-P) = \{0\}$. A conic preprime is simply a positive cone and induces a partial order: $x \geqq y \Leftrightarrow y \leqq x \Leftrightarrow x - y \in P$. A preprime P is *Archimedean* if for all x in K there exists a natural number n with $n - x$ in P, (condition (2) in the definition of Stone ring) and is (AC) if from $1 + nx \in P$ for all $n \in N$ follows $x \in P$ (condition (3)). We redefine a *Stone ring* as a pair $\langle K, P\rangle$ where P is an infinite conic Archimedean (AC) preprime in K. An *imbedding* of $\langle K, P\rangle$ in $\langle K', P'\rangle$ is an injective ring homomorphism $\psi: K \rightarrow K'$ such that $P = \psi^{-1}(P')$. If X is a compact Hausdorff space, $C(X)$ denotes the ring of all continuous real functions on X, $P(X)$ denotes the subset of nonnegative functions. If K is any subring of $C(X)$ then $\langle K, K \cap P(X)\rangle$ is a Stone ring. The principal tool in the proof of Theorem 1 is the Stone-Kadison ordered algebra theorem [3; Theorem 3.1], which characterizes $C(X)$ as a complete Archimedean ordered algebra. To imbed a Stone ring $\langle K, P\rangle$ in such an algebra we show that K is torsionfree, imbed it in a divisible ring K_N, put a norm on K_N and then complete it to K^*. At each step we have an imbedding of Stone rings: