PACIFIC JOURNAL OF MATHEMATICS
Vol. 21, No. 2, 1967

MACDONALD'S THEOREM WITH INVERSES

KEVIN MCCRIMMON

One of the fundamental theorems in the theory of Jordan algebras is that of I. G. Macdonald which says that any identity in three variables x, y, z of degree zero or one in z will be valid in all Jordan algebras if it is valid in the special Jordan algebras.

In this paper we will extend this result to identities which also involve the inverses of x and y.

Following the method and notation of N. Jacobson [3] we have the

THEOREM. *If $\mathfrak{J}$ and $\mathfrak{J}_s$ are respectively the free Jordan algebra and free special Jordan algebra on three free generators x, y, z and the inverses x^{-1}, y^{-1}, with $\mathfrak{E}$ and $\mathfrak{E}_s$ the associative algebras of linear transformations in $\mathfrak{J}$ and $\mathfrak{J}_s$ respectively generated by the multiplications by elements of the subalgebra generated by x, y, x^{-1}, y^{-1}, then the canonical homomorphism ν of $\mathfrak{E}$ onto $\mathfrak{E}_s$ is an isomorphism. If $\mathfrak{F}$ is the free associative algebra with free generators $f_{i,j} (i, j \in Z)$ and π the homomorphism of $\mathfrak{F}$ onto $\mathfrak{E}$ determined by $f_{i,j} \to U_{x^i, y^j}$ then the kernel of π is the ideal $\mathfrak{R}$ generated by the elements*

$$
(1)\qquad
\begin{array}{ll}
\text{(i)} & f_{0,0} - 1 \\
\text{(ii)} & 2f_{i,0}f_{j,k} - (2f_{i,0}^2 - f_{2i,0})f_{j-i,k} - f_{i+j,k} \\
 & 2f_{0,i}f_{k,j} - (2f_{0,i}^2 - f_{0,2i})f_{k,j-i} - f_{k,i+j} \\
\text{(iii)} & 2f_{j,k}f_{i,0} - f_{j-i,k}(2f_{i,0}^2 - f_{2i,0}) - f_{i+j,k} \\
 & 2f_{k,j}f_{0,i} - f_{k,j-i}(2f_{0,i}^2 - f_{0,2i}) - f_{k,i+j} .
\end{array}
$$

From this as immediate corollaries we have

MACDONALD'S THEOREM WITH INVERSES [4]. *If $\mathfrak{J}$ and $\mathfrak{J}_s$ are the free Jordan algebra and free special Jordan algebra on three free generators x, y, z and the inverses x^{-1}, y^{-1} then the kernel of the canonical homomorphism ν of $\mathfrak{J}$ onto $\mathfrak{J}_s$ contains no elements of degree zero or one in z.*

SHIRSHOV'S THEOREM WITH INVERSES [6]. *The free Jordan algebra on two free generators x, y and their inverses x^{-1}, y^{-1} is special.*

More generally, we have the

SHIRSHOV-COHN THEOREM WITH INVERSES [1]. *Any Jordan algebra*