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THE PTOLEMAIC INEQUALITY IN HILBERT GEOMETRIES

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Let M be a metric space, and if x and y are points in M, let xy denote the metric. The space M and its metric are called *ptolemaic* if for each quadruple of points x_i $(i = 1, 2, 3, 4)$ the *ptolemaic inequality*

$$x_1x_2 \cdot x_3x_4 + x_1x_3 \cdot x_2x_4 \geqq x_1x_4 \cdot x_2x_3$$

holds. If the inequality holds only in some neighborhood of each point the space and its metric are said to be *locally ptolemaic*. Euclidean space is known to be ptolemaic and therefore, locally ptolemaic. We are interested here in certain non-euclidean spaces which may possibly be locally ptolemaic. The author has proved in his thesis (Michigan State University Doctoral Dissertation, 1963) that a Riemannian geometry is locally ptolemaic if and only if it has nonpositive curvature, and that a Finsler space which is locally ptolemaic is Riemannian. The main result established here extends the theorem regarding Finsler spaces to include Hilbert geometries as well: A Hilbert geometry is locally ptolemaic if and only if it is hyperbolic.

The ptolemaic inequality is related to problems of curvature in metric geometry. Assuming this condition enables one to prove that a curve is a geodesic if and only if its metric curvature is zero at each of its points (see [3]). Blumenthal has investigated a number of properties peculiar to ptolemaic metric spaces in [2]. It is then significant to determine what metric spaces are ptolemaic. A question which remains unsettled is whether a non-Riemannian G-space (Busemann [4, p. 37]) can be locally ptolemaic. The result obtained here concerning Hilbert geometries, together with several in the author's thesis lends support to the conjecture that such a space does not exist.

Hilbert geometry is a generalization of the well-known Klein model for hyperbolic geometry. Consider an arbitrary bounded convex body C with nonempty interior D in euclidean space. If x and y be any two points in D, a distance function may be defined as

$$h(x, y) = k \mid \log R(xy, ab) \mid$$

where k is a positive constant, a and b are the points of intersection of C with the affine line L_{xy} determined by x and y, and $R(xy, ab)$ is