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ON OPERATORS WHOSE FREDHOLM SET IS THE COMPLEX PLANE

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Let T be a closed linear operator with domain and range in a complex Banach space X. The Fredholm set $\Phi(T)$ of T is the set of complex numbers λ such that $\lambda - T$ is a Fredholm operator. If the space X is of finite dimension then, obviously, the domain of T is closed and $\Phi(T)$ is the whole complex plane C. In this paper it is shown that the converse is also true. When T is defined on all of X this is a well-known result due to Gohberg and Kreĭn.

Examples of nontrivial closed operators with $\Phi(T) = C$ are the operators whose resolvent operator is compact. A characterization of the class of closed linear operators with a nonempty resolvent set and a Fredholm set equal to the complex plane will be given.

Throughout the present paper X and Y will denote complex Banach spaces. Let T be an arbitrary closed linear operator with domain $\mathscr{D}(T)$ in X and range $\mathscr{R}(T)$ in Y. The *nullity* $n(T)$ of T is the dimension of the null space $\mathscr{N}(T)$ of T. The *defect* $d(T)$ of T is the dimension of the quotient space $Y/\mathscr{R}(T)$. No distinction is made between infinite dimensions, so that $n(T)$ and $d(T)$ may be nonnegative integers or $+\infty$. We say that T is *Fredholm* if $n(T)$ and $d(T)$ are both finite. Note that $d(T) < \infty$ implies $\mathscr{R}(T)$ is closed (cf. [5], Lemma 332).

In 1957 Gohberg and Kreĭn [3] showed that if A is a bounded linear operator on X with $\Phi(A) = \boldsymbol{C}$, then the dimension of X (denoted by $\dim X$) is finite. The following theorem extends this result.

THEOREM 1. *Let T and S be bounded linear operators from X into Y. Suppose that S is a homeomorphism, and that $T + \lambda S$ is Fredholm for each $\lambda \in \boldsymbol{C}$. Then*

$$\dim X \leqq \dim Y < \infty .$$

Proof. Since S is a homeomorphism, $\mathscr{R}(S)$ is closed and $n(S) = 0$. By a well-known stability theorem (cf. [5], Theorem 1), this implies the existence of a positive constant ρ such that for $0 < |\mu| < \rho$

$$d(S) = d(S) - n(S) = d(S + \mu T) - n(S + \mu T) .$$

The right-hand side is finite because $S + \mu T$ is Fredholm for $\mu \neq 0$. Hence $d(S) < \infty$, and so S has a bounded left inverse, say R. Then $n(R) \leqq d(S) < \infty$ and $d(R) = 0$, so R is Fredholm. Define $A = RT$.