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AN ELEMENTARY PROOF THAT HAAR MEASURABLE ALMOST PERIODIC FUNCTIONS ARE CONTINUOUS

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It is known that a Haar measurable complex-valued (von Neumann) almost periodic function on a locally compact T_0-topological group is continuous. For by applying the Bohr-von Neumann approximation theorem for almost periodic functions and the fact that a Haar measurable representation into the general linear group is necessarily continuous one may deduce that such a function is the uniform limit of a sequence of continuous functions. This approach, while straightforward, has the disadvantage of depending on the very deep Bohr-von Neumann approximation theorem. The latter result is commonly proven through considerable usage of representation theory. This paper presents an alternative proof that Haar measurability plus almost periodicity imply continuity. The proof is elementary in the sense that it uses only the basic definitions of almost periodic function theory and topology. It does, however, depend on the standard tools of measure theory.

Suppose G is a locally compact T_0-topological group ($=LC$ group). Let Γ denote the set of Borel subsets of G, that is, the σ-algebra generated by the closed subsets of G. Let μ be a left Haar measure defined on Γ (cf. [2], pp. 184–215) and let $\bar{\Gamma}$ be the completion of Γ, that is, $\bar{\Gamma} = \{B \cup N\colon B \in \Gamma, N \subset N', \text{ where } N' \in \Gamma \text{ and } \mu N' = 0\}$. We extend μ to the σ-algebra $\bar{\Gamma}$ by defining $\mu(B \cup N) = \mu(B)$ for all $B \cup N \in \bar{\Gamma}$. μ, so extended, is left-invariant and regular on $\bar{\Gamma}$. By a $\bar{\Gamma}$-measurable function on G we mean a function f from G to the complex plane C such that $f^{-1}(A) \in \bar{\Gamma}$ for all Borel sets $A \subset C$. We are concerned with $\bar{\Gamma}$-measurable, rather Γ-measurable, functions so that in the real case, for example, we can deduce that Lebesgue measurable, as well as Borel measurable, almost periodic functions are continuous.

A set $A \subset G$ is called *bounded* if $\bar{A}$ is compact. We shall let e denote the identity of G and Ω the set of all bounded open neighborhoods of e in G. It is convenient to use the following "density theorem" whose proof is an exercise in Halmos' *Measure Theory* ([1], 61.5; Halmos' "Borel" sets are different from ours but his suggested proof works equally well in our setting.).

THEOREM. *Let G be an LC group. For any $U \in \Omega$, $x \in G$ and*