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# A REFINEMENT OF SELBERG'S ASYMPTOTIC EQUATION

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**The elementary proofs of the prime number theorem are essentially based on asymptotic equations of the form**

$$\text{(A)} \qquad f(x)\log x + \int_1^x f\left(\frac{x}{t}\right) d\psi(t) = O(x)\,,$$

**where $f(x)$ is some function concerning the primes, $\psi(x)$ is Tchebychev's function and the limits in the integral—as throughout in this paper—are taken from 1— to $x+$. This paper gives an elementary method for refining the right hand side of (A).**

**This method is based on the lemma of Tatuzawa and Iseki [2], and, assuming the prime number theorem, on an estimation of remainder integral which is more accurate than earlier ones.**

Writing

$$\psi(x) = \sum_{n \leqq x} \Lambda(n) = \sum_{p^\nu \leqq x} \log p, \quad R(x) = \psi(x) - x$$

we have the two equivalent forms of Selberg's asymptotic equation

$$(1) \qquad R(x)\log x + \int_1^x R\left(\frac{x}{t}\right) d\psi(t) = O(x)\,,$$

$$(2) \qquad \psi(x)\log x + \int_1^x \psi\left(\frac{x}{t}\right) d\psi(t) = 2x\log x + O(x)\,,$$

each of which is known to imply the prime number theorem: $\psi(x) = x + o(x)$. In this paper we give refinements of (1) and (2), showing that

$$(1') \qquad R(x)\log x + \int_1^x R\left(\frac{x}{t}\right) d\psi(t) = -(\gamma + 1)x + o(x)\,,$$

$$(2') \qquad \psi(x)\log x + \int_1^x \psi\left(\frac{x}{t}\right) d\psi(t) = 2x\log x - (2\gamma + 1)x + o(x)\,,$$

where $\gamma$ denotes Euler's constant. The prime number theorem, however, has then to be assumed. In addition, we give some similar results.

2. Using the idea of Tatuzawa and Iseki [2] we start from the following