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NONCONSTANT LOCALLY RECURRENT FUNCTIONS

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The purpose of this paper is to develop a new method of using the Baire Category Theorem to obtain counterexamples in analysis. The method is used to show that a certain class of nonconstant locally recurrent functions is of second category in a suitable metric space of continuous functions. In § 1 an explicit example is given of a nonconstant locally recurrent function. This example is included because it clarifies the category argument in § 4.

1. A simple example of a nonconstant locally recurrent function.

DEFINITION 1. A real-valued continuous function f of a real variable is said to be locally recurrent if for any x in its domain of definition and any neighborhood N of x, there exists $y \neq x$ in N such that $f(x) = f(y)$.

K. A. Bush [2] has given an example of a nonconstant locally recurrent function. The author believes that the example given below is simpler.

A sequence $\{f_n\}_{n=0}^{\infty}$ of functions on $[0, 1]$ will be defined. These functions are continuous and piecewise linear. Further, f_m is linear in any interval of the form $[n/9^m, (n+1)/9^m]$ where n and m are non-negative integers such that $0 \leqq n < 9^m$. Thus the function f_m is described completely if we give the values of $f_m(n/9^m)$, $(0 \leqq n \leqq 9^m)$. These functions will be defined inductively. We start with $f_0(x) = x$. Now suppose f_k is defined for some k. We define f_{k+1} as follows:

(a) $f_{k+1}(3m/9^{k+1}) = f_k(3m/9^{k+1}), 0 \leqq 3m \leqq 9^{k+1}$.

(b) $f_{k+1}((3m+1)/9^{k+1}) = f_k((3m+3)/9^{k+1}), 0 \leqq 3m+1, 3m+3 \leqq 9^{k+1}$.

(c) $f_{k+1}((3m+2)/9^{k+1}) = f_k(3m/9^{k+1}), 0 \leqq 3m, 3m+2 \leqq 9^{k+1}$.

The figure shows a portion of the graphs of f_k and f_{k+1}.

An important feature of these functions is the relation $f_k(n/9^k) = f_l(n/9^k)$ for $l \geqq k$ and $0 \leqq n \leqq 9^k$. Also notice that on any interval of the form $[n/9^k, (n+1)/9^k]$, the values of f_m, $m \geqq k$ must lie between $f_k(n/9^k)$ and $f_k((n+1)/9^k)$. It is not hard to see that the f_n converge uniformly and thus the limit function $f(x) = \lim_{n\to\infty} f_n(x)$ is continuous. It is obviously locally recurrent at points of the form $n/9^k$. That f is locally recurrent at any point x in $[0, 1]$ follows from an application of the intermediate value theorem for continuous functions. In fact, for any $k \geqq 0$, x must lie in an interval of the form $I = [3m/9^k,$